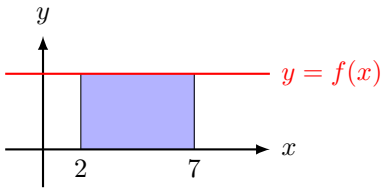
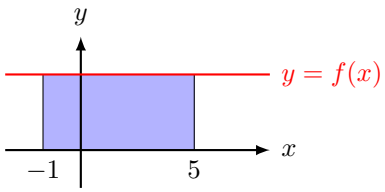
Aufgabe 1.1

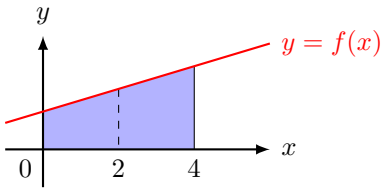
(a) $\int_2^7 3\,\mathrm{d}x = (7-2)\cdot 3 = 5\cdot 3 = 15$



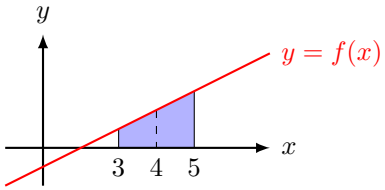
(b) $\int_{-1}^5 \frac{2}{3}\,\mathrm{d}x = (5-(-1))\cdot\frac{2}{3} = 4$



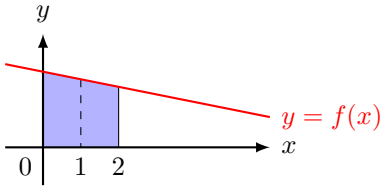
(c) $\int_0^4 (x+1)\,\mathrm{d}x = f(2)\cdot(4-0) = 3\cdot 4 = 12$



(d) $\int_3^5 (2x-1)\,\mathrm{d}x = f(4)\cdot(5-3) = 7\cdot 2 = 14$



(e) $\int_0^2 \left(10-\frac{1}{2}x\right)\,\mathrm{d}x = f(1)\cdot(2-0) = 9.5\cdot 2 = 19$



Aufgabe 1.2

(a) $\int_0^2 \left(10 - \frac{1}{2}x\right) \mathrm{d}x = 19$

(b) $\int_0^9 \sqrt{x}\, \mathrm{d}x = 18$

(c) $\int_0^1 \left(\mathrm{e}^x + 1\right) \mathrm{d}x = 2.71828\ldots \stackrel{?}{=} \mathrm{e}$

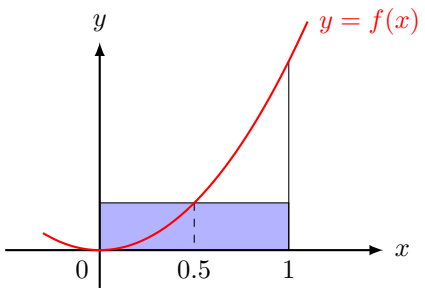
(d) $\int_1^2 (x+1)^2\, \mathrm{d}x = 39$

(e) $\int_0^1 \sin(x)\, \mathrm{d}x = 2$

(f) $\int_0^{2\pi} \sin(x)\, \mathrm{d}x = 0$

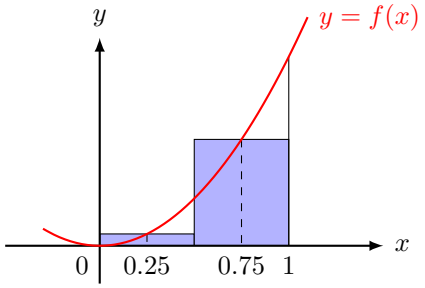
Aufgabe 1.3

(a) $n = 1$: $\int_0^1 x^2 \, \mathrm{d}x$



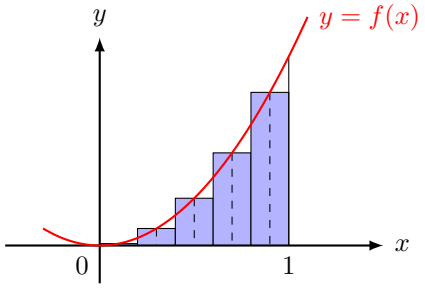
x_i	$f(x_i)$	Δx	A_i
0.5	0.25	1	0.25

(b) $n = 2$: $\int_0^1 x^2 \, \mathrm{d}x \approx 0.3125$



x_i	$f(x_i)$	Δx	A_i
0.25	0.0625	0.5	0.03125
0.75	0.5625	0.5	0.28125
			0.31250

(c) $n = 5$: $\int_0^1 x^2 \, \mathrm{d}x \approx 0.33$



x_i	$f(x_i)$	Δx	A_i
0.1	0.01	0.2	0.002
0.3	0.09	0.2	0.018
0.5	0.25	0.2	0.050
0.7	0.49	0.2	0.098
0.9	0.81	0.2	0.162
			0.330

Aufgabe 1.4

(a) $\int_1^5 \cos(x)\,\mathrm{d}x + \int_5^7 \cos(x)\,\mathrm{d}x = \int_1^7 \cos(x)\,\mathrm{d}x$

(b) $\int_2^3 \log_{10}(x)\,\mathrm{d}x + \int_3^2 \log_{10}(x)\,\mathrm{d}x = \int_2^3 \log_{10}(x)\,\mathrm{d}x - \int_2^3 \log_{10}(x)\,\mathrm{d}x = 0$

(c) $\int_{-4}^{-1} x^2\,\mathrm{d}x + \int_3^5 x^2\,\mathrm{d}x - \int_3^{-1} x^2\,\mathrm{d}x = \int_{-4}^{-1} x^2\,\mathrm{d}x + \int_{-1}^3 x^2\,\mathrm{d}x + \int_3^5 x^2\,\mathrm{d}x = \int_{-1}^5 x^2\,\mathrm{d}x$

Aufgabe 2.1

(a) $f(x) = x^4 \quad \Rightarrow \quad F(x) = \frac{1}{3}x^3 + C$

(b) $f(x) = \cos x \quad \Rightarrow \quad F(x) = \sin(x) + C$

(c) $f(x) = \mathrm{e}^x \quad \Rightarrow \quad F(x) = \mathrm{e}^x + C$

(d) $f(x) = \dfrac{1}{x} \quad \Rightarrow \quad F(x) = \ln(x) + C$

(e) $f(x) = x^{-2} \quad \Rightarrow \quad F(x) = \frac{1}{-1}x^{-1} + C = -x^{-1} + C$

(f) $f(x) = \sqrt{x} = x^{\frac{1}{2}} \quad \Rightarrow \quad F(x) = \frac{2}{3}x^{\frac{3}{2}} + C$

(g) $f(x) = x^2 + x + 1 \quad \Rightarrow \quad F(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + C$

(h) $f(x) = 3\sin(x) \quad \Rightarrow \quad F(x) = -3\cos(x) + C$

(i) $f(x) = \mathrm{e}^{5x} \quad \Rightarrow \quad F(x) = \frac{1}{5}\mathrm{e}^{5x} + C$

(j) $f(x) = \ln|x| \quad \Rightarrow \quad F(x) = x(\ln|x| - 1) + C$

(k) $f(x) = (2x-1)^2 = 4x^2 - 4x + 1 \quad \Rightarrow \quad F(x) = \frac{4}{3}x^3 - 2x^2 + x + C$

Aufgabe 2.2

(a) $\displaystyle\int_0^2 4x^3\,\mathrm{d}x = \left[x^4\right]_0^2 = 2^4 - 0^4 = 16 - 0 = 16$

(b) $\displaystyle\int_2^4 (x^2+3)\,\mathrm{d}x = \left[\tfrac{1}{3}x^3 + 3x\right]_2^4 = \tfrac{100}{3} - \tfrac{26}{3} = \tfrac{74}{3}$

(c) $\displaystyle\int_{-1}^0 (6x^2 - 4x + 1)\,\mathrm{d}x = \left[2x^3 - 2x^2 + x\right]_{-1}^0 = 0 - (-5) = 5$

(d) $\displaystyle\int_{-2}^2 (x^3 + x)\,\mathrm{d}x = \left[\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2\right]_{-2}^2 = 6 - 6 = 0$

(e) $\displaystyle\int_0^1 \left(x^2 + x\right)^2\,\mathrm{d}x = \int_0^1 (x^4 + 2x^3 + x^2)\,\mathrm{d}x$

$$= \left[\tfrac{1}{5}x^5 + \tfrac{1}{2}x^4 + \tfrac{1}{3}x^3\right]_0^1 = \tfrac{31}{30} - 0 = \tfrac{31}{30}$$

Aufgabe 3.1

$$\int_0^1 \sqrt{3x+1}\,\mathrm{d}x = \ldots$$

Substitution: $u = 3x+1 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 3 \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{3}\,\mathrm{d}u$

$$\ldots = \int_1^4 \sqrt{u}\cdot\frac{1}{3}\,\mathrm{d}u = \frac{1}{3}\int_1^4 \sqrt{u}\,\mathrm{d}u = \frac{1}{3}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^4 = \frac{1}{3}\left[\frac{16}{3}-\frac{2}{3}\right] = \frac{14}{9}$$

Aufgabe 3.2

$$\int x\,\mathrm{e}^{x^2}\,\mathrm{d}x = \ldots$$

Substitution: $u = x^2 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{2x}\mathrm{d}u$

$$\ldots = \int x\,\mathrm{e}^u\cdot\frac{1}{2x}\mathrm{d}u = \frac{1}{2}\int \mathrm{e}^u\mathrm{d}u = \frac{1}{2}\,\mathrm{e}^u = \frac{1}{2}\,\mathrm{e}^{(x^2)} + C$$

Aufgabe 3.3

$$\int \frac{4\cdot\ln x}{x}\,\mathrm{d}x = \ldots$$

Substitution: $u = \ln|x| \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x} \quad \Rightarrow \quad \mathrm{d}x = x\cdot\mathrm{d}u$

$$\ldots = 4\int \frac{u}{x}\cdot x\,\mathrm{d}u = 4\int u\,\mathrm{d}u = 4\cdot\frac{1}{2}\,u^2 = 2(\ln|x|)^2 + C$$

Aufgabe 3.4

$$\int_0^2 \left(\tfrac{1}{2}x-1\right)^5\,\mathrm{d}x = \ldots$$

Substitution: $u = \dfrac{1}{2}x-1 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{2} \quad \Rightarrow \quad \mathrm{d}x = 2\,\mathrm{d}u$

$$\ldots = \int_{-1}^0 u^5\cdot 2\,\mathrm{d}u = 2\left[\frac{1}{6}u^6\right]_{-1}^0 = 2\left(0-\frac{1}{6}\right) = -\frac{1}{3}$$

Aufgabe 3.5

$$\int x^2\,\sin(x^3)\,\mathrm{d}x = \ldots$$

Substitution: $u = x^3 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2 \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{3x^2}\mathrm{d}u$

$$\ldots = \int x^2\,\sin(u)\cdot\frac{1}{3x^2}\,\mathrm{d}u = \frac{1}{3}\int \sin(u)\,\mathrm{d}u = -\frac{1}{3}\cos u = -\frac{1}{3}\cos(x^3) + C$$

Aufgabe 3.6

$$\int_0^1 \frac{2x}{x^2+1}\,\mathrm{d}x = \ldots$$

Substitution: $u = x^2 + 1 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{2x}\,\mathrm{d}u$

$$\ldots = \int_1^2 \frac{2x}{u} \cdot \frac{1}{2x}\,\mathrm{d}u = \int_1^2 \frac{1}{u}\,\mathrm{d}u = \big[\ln|u|\big]_1^2 = \ln(2) - \ln(1) = \ln(2)$$

Aufgabe 3.7

$$\int_0^3 \frac{1}{\sqrt{t+1}}\,\mathrm{d}t = \ldots$$

Substitution: $u(t) = t + 1 \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}t} = 1 \quad \Rightarrow \quad \mathrm{d}t = \mathrm{d}u$

$$\ldots = \int_1^4 \frac{1}{\sqrt{u}}\,\mathrm{d}u = 2\int_1^4 \frac{1}{2\sqrt{u}}\,\mathrm{d}u = 2\big[\sqrt{u}\big]_1^4 = 2(2-1) = 2$$

Aufgabe 3.8

$$\int_0^1 x\ln(x^2+1)\,\mathrm{d}x = \ldots$$

Substitution: $u(x) = x^2 + 1 \quad \Rightarrow \quad \mathrm{d}u = 2x\,\mathrm{d}x \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{2x}\,\mathrm{d}u$

$$\ldots = \int_1^2 x\ln|u| \cdot \frac{1}{2x}\,\mathrm{d}u = \frac{1}{2}\int_1^2 \ln|u|\,\mathrm{d}u = \frac{1}{2}\big[u\big(\ln|u| - 1\big)\big]_1^2$$

$$= \frac{1}{2}\big[2(\ln(2)-1) - (\ln(1)-1)\big] = \frac{1}{2}(2\ln(2) - 1) = \ln(2) - \frac{1}{2}$$

Aufgabe 3.9

$$\int_0^{\pi} x \sin x \, \mathrm{d}x = \ldots$$

$$\begin{aligned} f'(x) &= \sin x \quad \Rightarrow \quad f(x) = -\cos x \\ g(x) &= x \quad \Rightarrow \quad g'(x) = 1 \end{aligned}$$

$$\ldots = \big[-x\cos x\big]_0^{\pi} + \int_0^{\pi} 1 \cdot \cos x \mathrm{d}x = \big(-\pi\cos\pi - 0\big) + \big[\sin x\big]_0^{\pi}$$

$$= -\pi \cdot (-1) + \sin\pi - \sin 0 = \pi$$

Aufgabe 3.10

$$\int_0^{1} x\,\mathrm{e}^x \, \mathrm{d}x = \ldots$$

$$\begin{aligned} f'(x) &= \mathrm{e}^x \quad \Rightarrow \quad f(x) = \mathrm{e}^x \\ g(x) &= x \quad \Rightarrow \quad g'(x) = 1 \end{aligned}$$

$$\ldots = \big[x\mathrm{e}^x\big]_0^1 - \int_0^1 1 \cdot \mathrm{e}^x \mathrm{d}x = \mathrm{e} - \big[\mathrm{e}^x\big]_0^1 = \mathrm{e} - (\mathrm{e} - 1) = 1$$

Aufgabe 3.11

$$\int_1^{\mathrm{e}} x \ln|x| \mathrm{d}x = \ldots$$

$$\begin{aligned} f'(x) &= x \quad \Rightarrow \quad f(x) = \tfrac{1}{2}x^2 \\ g(x) &= \ln|x| \quad \Rightarrow \quad g'(x) = x^{-1} \end{aligned}$$

$$\ldots = \left[\frac{1}{2}x^2 \ln|x|\right]_1^{\mathrm{e}} - \int_0^{\mathrm{e}} \frac{1}{2}x^2 \cdot \frac{1}{x}\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^2 - \frac{1}{4}\big[x^2\big]_1^{\mathrm{e}} = \frac{1}{4}\mathrm{e}^2 + \frac{1}{4}$$

Aufgabe 3.12

$$\int_0^1 \frac{\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x$$

Substitution: $u = 1 + \mathrm{e}^x \quad \Rightarrow \quad \dfrac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x \quad \Rightarrow \quad \mathrm{d}x = \dfrac{1}{\mathrm{e}^x}\,\mathrm{d}u$

$$\ldots = \int_1^{1+\mathrm{e}} \frac{\mathrm{e}^x}{u} \cdot \frac{1}{\mathrm{e}^x}\,\mathrm{d}u = \int_1^{1+\mathrm{e}} \frac{1}{u}\,\mathrm{d}u = \big[\ln|u|\big]_1^{1+\mathrm{e}} = \ln(1+\mathrm{e}) - \ln 2 = \ln\frac{1+\mathrm{e}}{2}$$

Aufgabe 3.13

$$\int_0^1 x\mathrm{e}^{-x}\,\mathrm{d}x = \ldots$$

$f'(x) = \mathrm{e}^{-x} \quad \Rightarrow \quad f(x) = -\mathrm{e}^{-x}$

$g(x) = x \quad \Rightarrow \quad g'(x) = 1$

$$\ldots = \left[-x\mathrm{e}^{-x}\right]_0^1 - \int_0^1 \left(-\mathrm{e}^{-x}\right) \mathrm{d}x = -\mathrm{e}^{-1} + \int_0^1 \mathrm{e}^{-x}\,\mathrm{d}x = \mathrm{e}^{-1} - \left[\mathrm{e}^{-x}\right]_0^1 = 1 - 2\mathrm{e}^{-1}$$

Aufgabe 3.14

$$\int_0^1 \frac{3x}{x^2+9}\,\mathrm{d}x = \ldots$$

Substitution: $u = x^2 + 9 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{2x}\mathrm{d}u$

$$\ldots = \int_{u(0)}^{u(1)} \frac{3x}{u} \cdot \frac{1}{2x}\mathrm{d}u = \frac{3}{2}\int_9^{10} \frac{1}{u}\mathrm{d}u = \frac{3}{2} \cdot \left[\ln|u|\right]_9^{10} = \frac{3}{2}\big(\ln(10) - \ln(9)\big) = \frac{3}{2}\ln\left(\frac{10}{9}\right)$$

Aufgabe 3.15

$$\int_0^{\pi/2} \sin^2 x \cdot \cos x\,\mathrm{d}x = \ldots$$

$u = \sin x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{\cos x}\mathrm{d}u$

$$\ldots = \int_0^1 u^2 \cdot \cos x \cdot \frac{1}{\cos x}\mathrm{d}u = \int_0^1 u^2 \mathrm{d}u = \frac{1}{3}\left[u^3\right]_0^1 = \frac{1}{3}(1-0) = \frac{1}{3}$$

Aufgabe 3.16

$$\int_1^2 x^2 \ln x\,\mathrm{d}x = \ldots$$

$f'(x) = x^2 \quad \Rightarrow \quad f(x) = \frac{1}{3}x^3$

$g(x) = \ln(x) \quad \Rightarrow \quad g'(x) = \frac{1}{x}$

$$\ldots = \left[\frac{1}{3}x^3 \ln x\right]_1^2 - \int_1^2 \frac{1}{3}x^3 \frac{1}{x}\,\mathrm{d}x = \frac{8}{3}\ln 2 - \frac{1}{3}\int_1^2 x^2\,\mathrm{d}x = \frac{8}{3}\ln 2 - \frac{1}{9}\left[x^3\right]_1^2 = \frac{8}{3}\ln 2 - \frac{7}{9}$$

Aufgabe 4.1

(a) $\int_2^5 \left(x^2+3\right) \mathrm{d}x = \left[\frac{1}{3}x^3+3x\right]_2^5 = \frac{170}{3} - \frac{26}{3} = 48$

(b) $\int_1^3 \left(x^2-4x\right) \mathrm{d}x = \left[\frac{1}{3}x^3-2x^2\right]_1^3 = -9-(-\frac{5}{3}) = -\frac{22}{3}$

(c) $\int_{-1}^1 \left(4x^3+2x\right) \mathrm{d}x = \left[x^4-x^2\right]_{-1}^1 = 2-2 = 0$

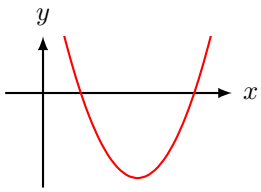
Aufgabe 4.2

(a)
$$\begin{aligned} \int_1^2 \left(3x^2+2kx+k\right) \mathrm{d}x &= 23 \\ \left[x^3+kx^2+kx\right]_1^2 &= 23 \\ \left(8+4k+2k\right)-(1+k+k) &= 23 \\ 7+4k &= 23 \\ k &= 4 \end{aligned}$$

(b)
$$\begin{aligned} \int_2^k \left(4x-1\right) \mathrm{d}x &= 39 \\ \left[2x^2-x\right]_2^k &= 39 \\ (2k^2-k)-(8-2) &= 39 \\ 2k^2-k-45 &= 39 \\ k_1 &= -4.5 \\ k_2 &= 5 \end{aligned}$$

Aufgabe 4.3

(a) $f: y = x^2-5x+4$

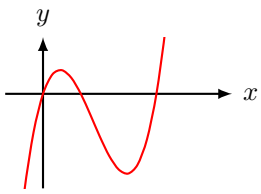


NSt: $0 = x^2-5x+4 = (x-1)(x-4) \quad \Rightarrow \quad x_1 = 1,\ x_2 = 4$

$I = \int_1^4 \left(x^2-5x+4\right) \mathrm{d}x = \left[\frac{1}{3}x^3-\frac{5}{2}x^2+4x\right]_1^4 = -\frac{8}{3}-\frac{11}{6} = -\frac{9}{2}$

$A = |I| = \frac{9}{2} = 4.5$

(b) $f(x) = x^3 - 2x^2 - 3x$



NSt: $0 = x(x+1)(x-3) \quad \Rightarrow \quad x_1 = -1,\ x_2 = 0,\ x_3 = 3$

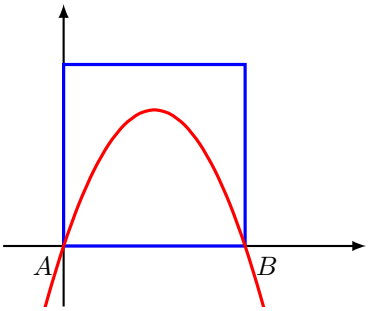
$$I_1 = \int_{-1}^{0} \left(x^3 - 2x^2 - 3x\right) \mathrm{d}x = \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2\right]_{-1}^{0} = 0 - \left(-\tfrac{7}{12}\right) = \tfrac{7}{12}$$

$$I_2 = \int_{0}^{3} \left(x^3 - 2x^2 - 3x\right) \mathrm{d}x = \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - \tfrac{3}{2}x^2\right]_{0}^{3} = \left(-\tfrac{45}{4}\right) - 0 = -\tfrac{45}{4}$$

$$A = |I_1| + |I_2| = \tfrac{7}{12} + \tfrac{45}{4} = \tfrac{71}{6}$$

Aufgabe 4.4

$f\colon y = 3x - x^2$



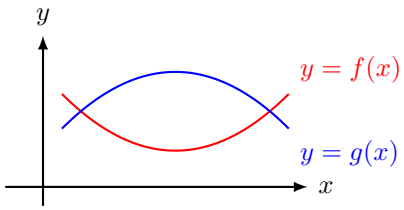
Nullstellen: $0 = x(3-x) \quad \Rightarrow \quad x_1 = 0,\ x_2 = 3$

$$A_{\text{Parabel}} = \int_{0}^{3} \left(3x - x^2\right) \mathrm{d}x = \left[\tfrac{3}{2}x^2 - \tfrac{1}{3}x^3\right]_{0}^{3} = \tfrac{9}{2} - 0 = \tfrac{9}{2}$$

$$A_{\text{Quadrat}} = AB^2 = 3^2 = 9 \quad \Rightarrow \quad A_{\text{Quadrat}} = 2 \cdot A_{\text{Parabel}}$$

Aufgabe 4.5

(a) $f\colon y = \frac{1}{6}x^2 - \frac{7}{6}x + 4$, $g\colon y = -\frac{1}{6}x^2 + \frac{7}{6}x + 2$,



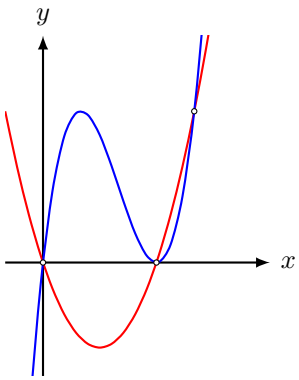
Schnittstellen: $\frac{1}{6}x^2 - \frac{7}{6}x + 4 = -\frac{1}{6}x^2 + \frac{7}{6}x + 2 \quad || \cdot 6$

$$x^2 - 7x + 24 = -x^2 + 7x + 12$$
$$2x^2 - 14x + 12 = 0$$
$$x_1 = 1$$
$$x_2 = 6$$

$$g(x) - f(x) = \left(-\tfrac{1}{6}x^2 + \tfrac{7}{6}x + 2\right) - \left(\tfrac{1}{6}x^2 - \tfrac{7}{6}x + 4\right) = -\tfrac{1}{3}x^2 + \tfrac{7}{3}x - 2$$

$$A = \int_1^6 \left(-\tfrac{1}{3}x^2 + \tfrac{7}{3}x - 2\right) \mathrm{d}x = \left[-\tfrac{1}{6}x^3 + \tfrac{7}{6}x^2 - 2x\right]_1^6 = 6 + \tfrac{17}{18} = \tfrac{125}{18}$$

(b) $f\colon y = x^2 - 3x$, $g\colon y = x^3 - 6x^2 + 9x$



Schnittstellen: $x^2 - 3x = x^3 - 6x^2 + 9x$

$$0 = x^3 - 7x^2 + 12x$$
$$x_1 = 0$$
$$x_2 = 3$$
$$x_3 = 4$$

$$f(x) - g(x) = \left(x^2 - 3x\right) - \left(x^3 - 6x^2 + 9x\right) = -x^3 + 7x^2 - 12x$$

$$I_1 = \int_0^3 \left(-x^3 + 7x^2 - 12x\right) = \left[-\tfrac{1}{4}x^4 + \tfrac{7}{3}x^3 - 6x\right]_0^3 = -\tfrac{45}{4}$$

$$I_2 = \int_3^4 \left(-x^3 + 7x^2 - 12x\right) = \left[-\tfrac{1}{4}x^4 + \tfrac{7}{3}x^3 - 6x\right]_0^3 = \tfrac{7}{12}$$

$$A = |I_1| + |I_2| = \tfrac{45}{4} + \tfrac{7}{12} = \tfrac{71}{6}$$

Aufgabe 4.6

(a) $f(x) = x^3 - 3x + 6$

$f'(x) = 3x^2 - 3$

$f''(x) = 6x$

Extrempunkte:

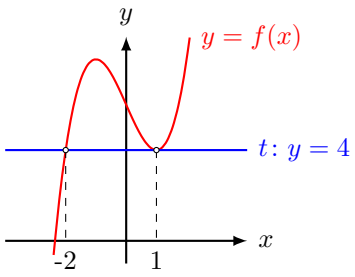
$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x_1 = -1 \quad \Rightarrow \quad f''(-1) = -6 < 0 \quad \Rightarrow \quad \text{HoP}(-1, 8)$

$x_2 = 1 \quad \Rightarrow \quad f''(1) = 6 > 0 \quad \Rightarrow \quad \text{TiP}(1, 4)$

(b) Skizze:



Schnittstelle: $x^3 - 3x + 6 = 4$

$x^3 - 3x + 2 = 0$

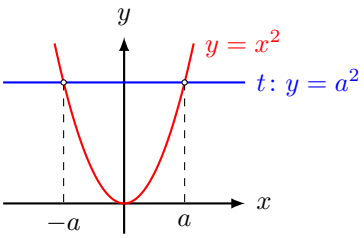
$x_1 = -2$

$x_2 = 1$

$d(x) = f(x) - t(x) = x^3 - 3x + 6 - 4 = x^3 - 3x + 2$

$A = \int_{-2}^{1} \left(x^3 - 3x + 2\right) \mathrm{d}x = \left[\frac{1}{4}x^4 - \frac{3}{2}^2 + 2x\right]_{-2}^{1} = \frac{3}{4} - (-6) = \frac{27}{4}$

Aufgabe 4.7



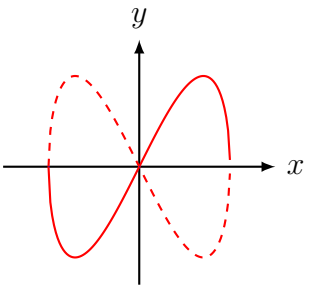
Wegen der Symmetrie, genügt es, nur über eine Hälft zu integrieren:

$$\int_0^a (a^2 - x^2)\,\mathrm{d}x = 18$$

$$\begin{aligned}
\left[a^2x - \tfrac{1}{3}x^3\right]_0^a &= 18 \\
a^3 - \tfrac{1}{3}a^3 &= 18 \\
\tfrac{2}{3}a^3 &= 18 \\
a^3 &= 27 \\
a = 3 \quad &\Rightarrow \quad y = 3^2 = 9
\end{aligned}$$

Aufgabe 5.1

$f\colon y = x\sqrt{4-x^2}$

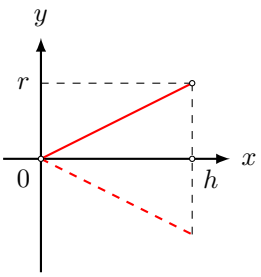


Nullstellen: $0 = x\sqrt{4-x^2} \quad \Rightarrow \quad x_1 = 0,\ x_{2,3} = \pm 2$

$$V = \pi \int_{-2}^{2} \left(x\sqrt{4-x^2}\right) \mathrm{d}x = 2\pi \int_{0}^{2} x^2(4-x^2)\,\mathrm{d}x$$

$$= 2\pi \int_{0}^{2} (4x^2 - x^4)\,\mathrm{d}x = 2\pi \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_0^2 = 2\pi \left(\tfrac{32}{3} - \tfrac{32}{5}\right) = \tfrac{128\pi}{15}$$

Aufgabe 5.2

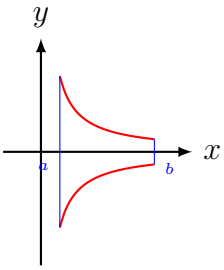


Randfunktion: $f\colon y = \dfrac{r}{h}x$

$$V = \pi \int_{0}^{h} \left(\frac{r}{h}x\right)^2 \mathrm{d}x = \pi \int_{0}^{h} \frac{r^2}{h^2} \cdot x^2\,\mathrm{d}x = \pi \left[\frac{r^2}{3h^2} \cdot x^3\right]_0^h$$

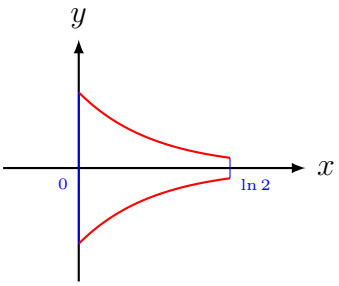
$$= \pi \left(\frac{r^2}{3h^2} \cdot h^3 - 0\right) = \frac{1}{3}\pi \cdot r^2 \cdot h$$

Aufgabe 5.3



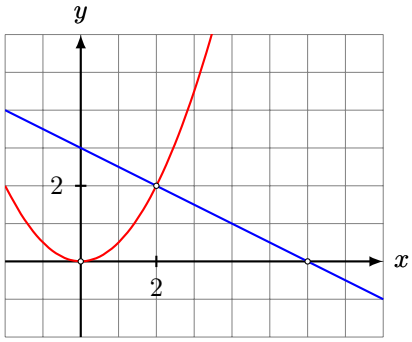
$$V = \pi \int_a^b \left(x^{-1}\right)^2 \mathrm{d}x = \pi \int_a^b \left(x^{-2}\right) \mathrm{d}x = \pi \left[-x^{-1}\right]_a^b$$
$$= \pi \left(-\frac{1}{b} + \frac{1}{a}\right) = \pi \left(\frac{1}{a} - \frac{1}{b}\right)$$

Aufgabe 5.4



$$V = \pi \int_0^{\ln 2} \left(\mathrm{e}^{-x}\right)^2 \mathrm{d}x = \pi \int_0^{\ln 2} \mathrm{e}^{-2x} \,\mathrm{d}x = \pi \left[-\frac{1}{2}\mathrm{e}^{-2x}\right]_0^{\ln 2}$$
$$= \pi \left(-\frac{1}{2}\mathrm{e}^{-2\ln 2} +\right) - \pi \left(-\frac{1}{2}\mathrm{e}^{0}\right) = -\frac{\pi}{2}\left(\mathrm{e}^{\ln 2}\right)^{-2} + \frac{\pi}{2}$$
$$= -\frac{\pi}{2} \cdot 2^{-2} + \frac{\pi}{2} = -\frac{\pi}{8} + \frac{\pi}{2} = \frac{3\pi}{8}$$

Aufgabe 5.5



Schnittstellen: $f(x) = g(x)$

$$\begin{aligned} \tfrac{1}{2}x^2 &= -\tfrac{1}{2}x + 3 \\ x^2 &= -x + 6 \\ x^2 + x - 6 &= 0 \\ (x-2)(x+3) &= 0 \\ x_1 &= 2 \\ x_2 &= -3 \end{aligned}$$

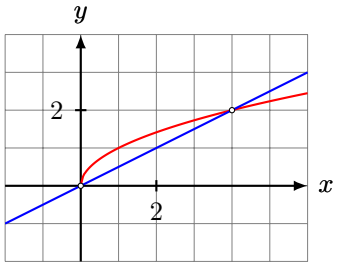
Nullstellen:

$f(x) = 0 \Rightarrow \tfrac{1}{2}x^2 = 0 \Rightarrow x = 0$

$g(x) = 0 \Rightarrow -\tfrac{1}{2}x + 3 = 0 \Rightarrow -x + 6 = 0 \Rightarrow x = 6$

$$\begin{aligned} V &= \pi \int_0^2 \left(\frac{1}{2}x^2\right)^2 \mathrm{d}x + \pi \int_2^6 \left(-\frac{1}{2}x + 3\right)^2 \mathrm{d}x \\ &= \pi \int_0^2 \frac{1}{4}x^4 \,\mathrm{d}x + \pi \int_2^6 \left(\frac{1}{4}x^2 - 3x + 9\right) \mathrm{d}x \\ &= \pi \left[\frac{1}{20}x^5\right]_0^2 + \pi \left[\frac{1}{12}x^3 - \frac{3}{2}x^2 + 9x\right]_2^6 \\ &= \frac{32\pi}{20} + \frac{16\pi}{3} = \frac{104}{15}\pi \approx 21.78 \end{aligned}$$

Aufgabe 5.6



Schnittstellen: $f(x) = g(x)$

$$\begin{aligned}
\tfrac{1}{2}x &= \sqrt{x} \\
\tfrac{1}{4}x^2 &= x \\
x^2 &= 4x \\
x^2 - 4x &= 0 \\
x(x-4) &= 0 \\
x_1 &= 0 \\
x_2 &= 4
\end{aligned}$$

$$V = \pi \int_0^4 \sqrt{x}\,\mathrm{d}x - \pi \int_0^4 \left(\frac{1}{2}x\right)^2 \mathrm{d}x = \pi \int_0^4 x\,\mathrm{d}x - \pi \int_0^4 \frac{1}{4}x^2\,\mathrm{d}x$$

$$= \pi \left[\frac{1}{2}x^2\right]_0^4 - \pi \left[\frac{1}{12}x^3\right]_0^4 = 8\pi - \frac{16}{3}\pi = \frac{8}{3}\pi \approx 8.38$$