Formel von Taylor

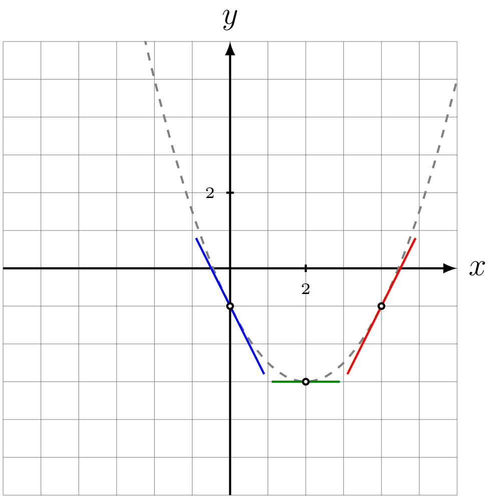
$$T_n f(x; x_0) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Aufgabe 1

$f(x) = \frac{1}{2}x^2 - 2x - 1$

$f'(x) = x - 2$

- $T_1 f(x; 0) = f(0) + f'(0)(x - 0) = -1 - 2x$
- $T_1 f(x; 2) = f(2) + f'(0)(x - 2) = -3 + 0(x - 2)$
- $T_1 f(x; 4) = f(4) + f'(4)(x - 4) = -1 + 2(x - 4)$



Aufgabe 2

$f(x) = \frac{1}{3}x^3 - x^2 - 3x + 2$

$f'(x) = x^2 - 2x - 3$

$f''(x) = 2x - 2$

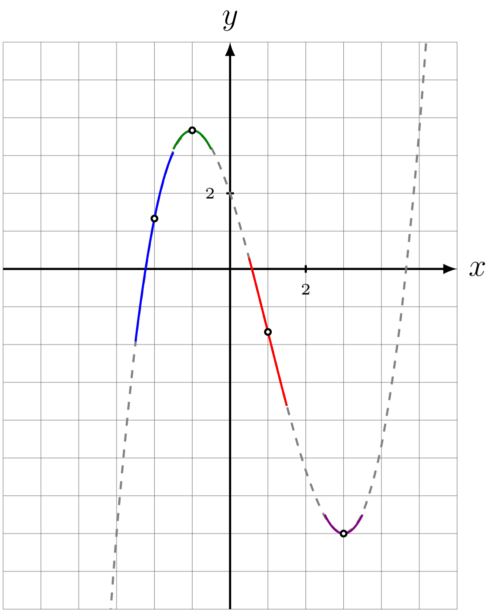
$f'''(x) = 2$

- $$\begin{aligned} T_2 f(x; -2) &= f(-2) + f'(-2)(x+2) + \tfrac{1}{2} f''(-2)(x+2)^2 \\ &= \tfrac{4}{3} + 5(x+2) - 6(x+2)^2 \end{aligned}$$

- $$\begin{aligned} T_2 f(x; -1) &= f(-1) + f'(-1)(x+1) + \tfrac{1}{2} f''(-1)(x+1)^2 \\ &= \tfrac{11}{3} + 0(x+1) - 4(x+1)^2 \end{aligned}$$

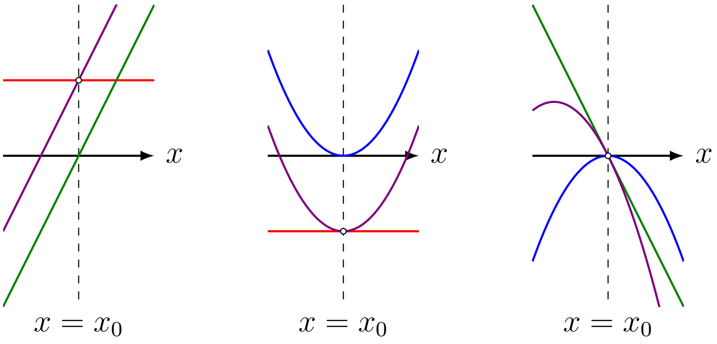
- $$\begin{aligned} T_3 f(x; 1) &= f(1) + f'(1)(x-1) + \tfrac{1}{2} f''(1)(x-1)^2 + \tfrac{1}{6} f'''(1)(x-1)^2 \\ &= -\tfrac{5}{3} - 4(x-1) - 0(x+1)^2 + \tfrac{1}{3}(x-1)^3 \end{aligned}$$

- $$\begin{aligned} T_2 f(x; 3) &= f(3) + f'(3)(x-3) + \tfrac{1}{2} f''(3)(x-3)^2 \\ &= -7 + 0(x-3) + 4(x-3)^2 \end{aligned}$$



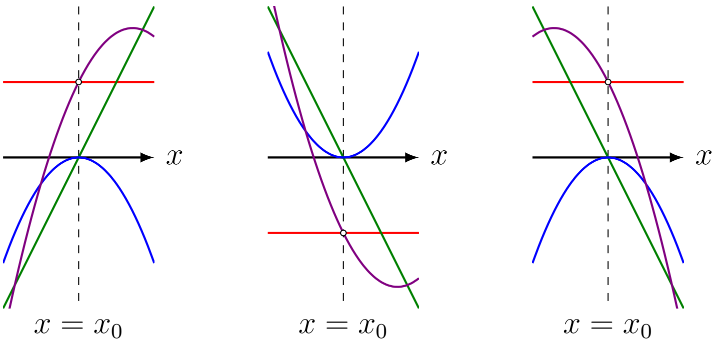
Aufgabe 3

Skizziere die Summe der Monome in der Umgebung von x_0.



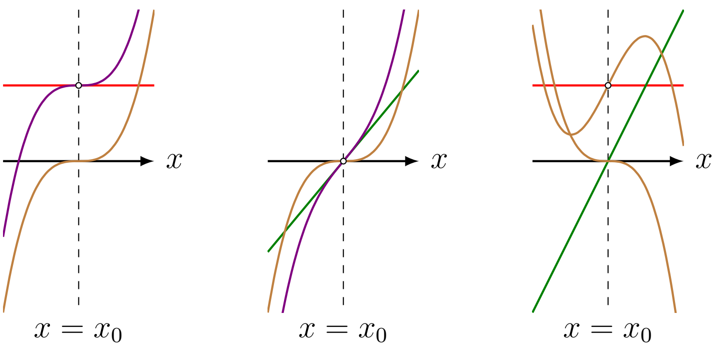
Aufgabe 4

Skizziere die Summe der Monome in der Umgebung von x_0.



Aufgabe 5

Skizziere die Summe der Monome in der Umgebung von x_0.



Aufgabe 6

Gegeben: $f(x) = \frac{1}{3}x^3 - 2x^2 + 3x + 1$

Ableitungen:
$$\begin{aligned} f'(x) &= x^2 - 4x + 3 \\ f''(x) &= 2x - 4 \\ f'''(x) &= 2 \end{aligned}$$

Asymptotisches Verhalten:
$$\begin{aligned} \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty} (x^3) = -\infty \\ \lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty} (x^3) = +\infty \end{aligned}$$

Nullstellen:
$$\begin{aligned} f(x) &= 0 \\ \frac{1}{3}x^3 - 2x^2 + 3x + 1 &= 0 \\ x &= -0.279 \end{aligned}$$

Extrempunkte:
$$\begin{aligned} f'(x) &= 0 \\ x^2 - 4x + 3 &= 0 \\ x_1 &= 1 \quad \text{(Kandidaten)} \\ x_2 &= 3 \end{aligned}$$

x_0	a_3	-2	3	1
1	$\frac{1}{3}$	$-\frac{5}{3}$	$\frac{4}{3}$	$\frac{7}{3} = f(1)$
1	$\frac{1}{3}$	$-\frac{4}{3}$	$0 = f'(1)$	
1	$\frac{1}{3}$	$-1 = \frac{1}{2}f''(1)$		

$\Rightarrow$ HoP$(1, \frac{7}{3})$

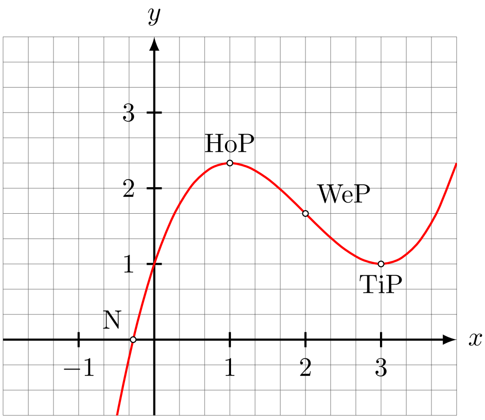
x_0	a_3	-2	3	1
3	$\frac{1}{3}$	-1	0	$1 = f(3)$
3	$\frac{1}{3}$	0	$0 = f'(3)$	
3	$\frac{1}{3}$	$1 = \frac{1}{2}f''(3)$		

$\Rightarrow$ TiP$(3, 1)$

Wendepunkte:
$$\begin{aligned} f''(x) &= 0 \\ 2x - 4 &= 0 \\ x &= 2 \quad \text{(Kandidat)} \end{aligned}$$

x_0	a_3	-2	3	1
2	$\frac{1}{3}$	$-\frac{4}{3}$	$\frac{1}{3}$	$\frac{5}{3} = f(2)$
2	$\frac{1}{3}$	$-\frac{2}{3}$	$-1 = f'(2)$	
2	$\frac{1}{3}$	$0 = \frac{1}{2}f''(2)$		
2	$\frac{1}{3} = \frac{1}{6}f'''(2)$			

$\Rightarrow$ WeP$(2, \frac{5}{3})$


y
x
3
2
1
−1
1
2
3
HoP
WeP
TiP
N

Aufgabe 7

$$\begin{aligned} f(x) &= f(x) = x^4 - 4x^3 = x^3(x-4) \\ f'(x) &= 4x^3 - 12x^2 = 4x^2(x-3) \\ f''(x) &= 12x^2 - 24x = 12x(x-2) \\ f'''(x) &= 24x - 24 = 24(x-1) \end{aligned}$$

Asymptotisches Verhalten: $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x^4) = \infty$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} (x^4) = \infty$

Nullstellen: $$\begin{aligned} f(x) &= 0 \\ x^3(x-4) &= 0 \\ x_1 &= 0 \\ x_2 &= 4 \end{aligned}$$

Extrempunkte: $$\begin{aligned} f'(x) &= 0 \\ 4x^2(x-3) &= 0 \\ x_1 &= 0 \quad \text{(Kandidaten)} \\ x_2 &= 3 \end{aligned}$$

$x = 0$: $f(0) = 0,\ f'(0) = 0,\ f''(0) = 0,\ f'''(0) = -24$

$\Rightarrow$ TeP(0,0)

$x = 3$: $f(3) = -27,\ f'(3) = 0,\ f''(3) = 36 > 0$

$\Rightarrow$ TiP(3,-27)

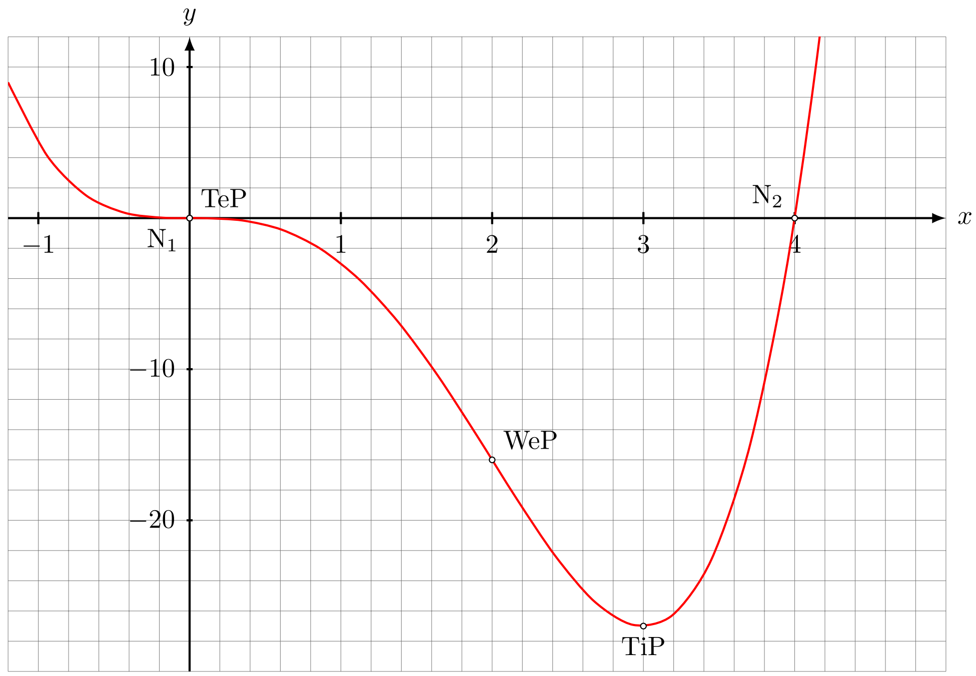
Wendepunkte: $$\begin{aligned} f''(x) &= 0 \\ 12x(x-2) &= 0 \\ x_1 &= 0 \quad \text{(wurde bereits untersucht)} \\ x_2 &= 2 \quad \text{(Kandidat)} \end{aligned}$$

$x = 2$: $f(2) = 0,\ f'(0) = -16,\ f''(0) = 0,\ f'''(0) = 24 > 0$

$\Rightarrow$ WeP(2,-16)

$x = 3$: $f(3) = -27,\ f'(3) = 0,\ f''(3) = 36 > 0$

$\Rightarrow$ TiP(3,-27)


y
10
−10
−20
x
−1
1
2
3
4
TeP
N$_1$
N$_2$
WeP
TiP

Aufgabe 8

$$\begin{aligned} f(x) &= x^4 - x = x(x^3-1) \\ f'(x) &= 4x^3 - 1 \\ f''(x) &= 12x^2 \\ f'''(x) &= 24x \\ f^{(4)}(x) &= 24 \end{aligned}$$

Asymptotisches Verhalten: $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x^4) = \infty$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} (x^4) = \infty$

Nullstellen:
$$\begin{aligned} f(x) &= 0 \\ x(x^3-1) &= 0 \\ x_1 &= 0 \\ x_2 &= 1 \end{aligned}$$

Extrempunkte:
$$\begin{aligned} f'(x) &= 0 \\ 4x^3 - 1 &= 0 \\ x &= \sqrt[3]{\tfrac{1}{4}} \approx 0.630 \end{aligned}$$

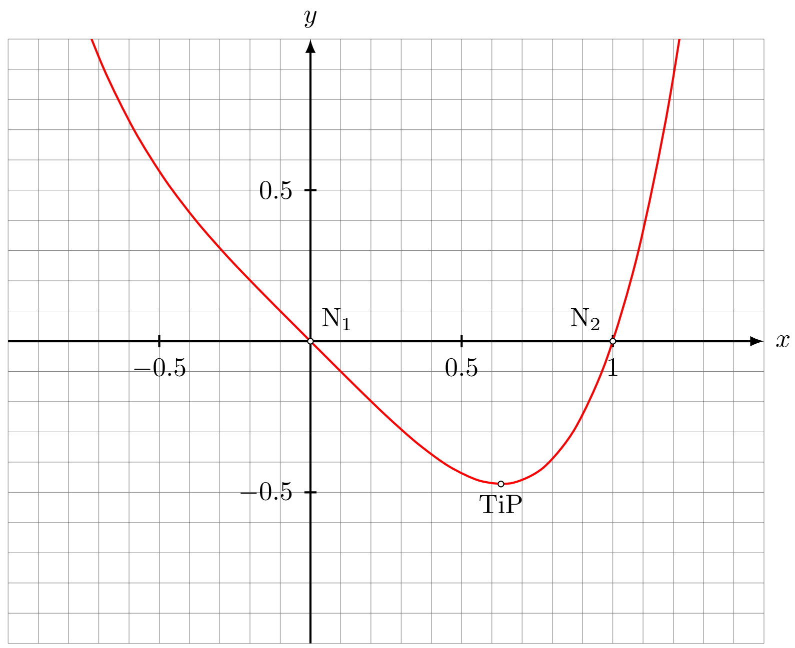
$x = \sqrt[3]{\frac{1}{4}}$: $f\left(\sqrt[3]{\frac{1}{4}}\right) = -0.472,\ f'\left(\sqrt[3]{\frac{1}{4}}\right) = 0,\ f''\left(\sqrt[3]{\frac{1}{4}}\right) = 4.762$

$\Rightarrow$ TiP$(0.630, -0.472)$

Wendepunkte:
$$\begin{aligned} f''(x) &= 0 \\ 12x^2 &= 0 \\ x &= 0 \quad \text{(Kandidat)} \end{aligned}$$

$x = 0$: $f(0) = 0,\ f'(0) = -1,\ f''(0) = f'''(0) = 0,\ f^{(4)}(0) = 24$

$\Rightarrow (0,0)$ ist kein Wendepunkt (Kurve fallend und linksgekrümmt)



Aufgabe 9

$$f(x) = \frac{1}{2}x^4 - 4x^2 + 6$$
$$f'(x) = 2x^3 - 8x$$
$$f''(x) = 6x^2 - 8$$
$$f'''(x) = 12x$$

Asymptotisches Verhalten: $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x^4) = \infty$

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(x^4) = \infty$

Nullstellen:
$$f(x) = 0$$
$$\frac{1}{2}x^4 - 4x^2 + 6 = 0$$
$$x^4 - 8x^2 + 12 = 0$$
$$(x^2 - 2)(x^2 - 6) = 0$$
$$x_{1,2} = \pm\sqrt{2} \approx 1.414$$
$$x_{3,4} = \pm\sqrt{6} \approx 2.449$$

Extrempunkte:
$$f'(x) = 0$$
$$2x^3 - 8x = 0$$
$$2x(x^2 - 4) = 0$$
$$x_1 = 0$$
$$x_2 = -2$$
$$x_3 = 2$$

$x = 0$: $f(0) = 6,\ f'(0) = 0,\ f''(0) = -8$

$\Rightarrow$ HoP$(0, 6)$

$x = -2$: $f(-2) = -2,\ f'(0) = 0\ f''(-2) = 16$

$\Rightarrow$ TiP$_1(-2, -2)$

$x = 2$: $f(2) = 6,\ f'(0) = 0\ f''(2) = 16$

$\Rightarrow$ TiP$_2(2, -2)$

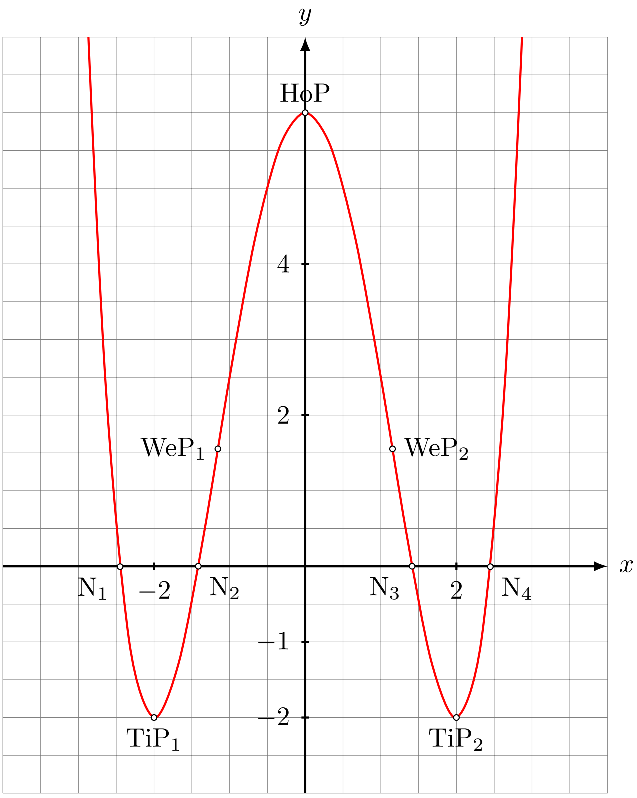
Wendepunkte:
$$f''(x) = 0$$
$$6x^2 - 8 = 0$$
$$x = \pm\sqrt{\frac{4}{3}} \approx \pm 1.16$$

$x = -\sqrt{\frac{4}{3}}$: $f\left(-\sqrt{\frac{4}{3}}\right) = 1.56,\ f'\left(-\sqrt{\frac{4}{3}}\right) = 6.16,\ f''\left(-\sqrt{\frac{4}{3}}\right) = 0,\ f'''\left(-\sqrt{\frac{4}{3}}\right) = -13.16$

$\Rightarrow$ WeP$_1(-1.16, 1.56)$ (LK $\rightarrow$ RK)

$x = \sqrt{\frac{4}{3}}$: $f\left(\sqrt{\frac{4}{3}}\right) = 1.56,\ f'\left(\sqrt{\frac{4}{3}}\right) = -6.16,\ f''\left(\sqrt{\frac{4}{3}}\right) = 0,\ f'''\left(\sqrt{\frac{4}{3}}\right) = 13.16$

$\Rightarrow$ WeP$_2(1.16, 1.56)$ (RK $\rightarrow$ LK)



y
HoP
4
2
WeP$_1$
WeP$_2$
x
N$_1$
−2
N$_2$
N$_3$
2
N$_4$
−1
−2
TiP$_1$
TiP$_2$