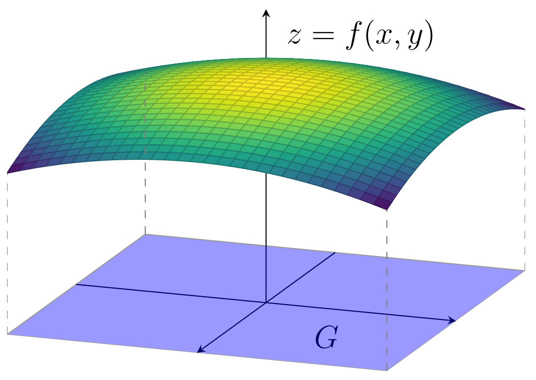
AM3 Gebietsintegrale (Integralrechnung: Kapitel 9)



$$V \overset{Def.}{=} \iint f(x,y)\mathrm{d}G = \lim_{\substack{\Delta G_i \to 0 \\ n \to \infty}} \sum_{i=1}^{n} f(x_i, y_i)\Delta G_i$$

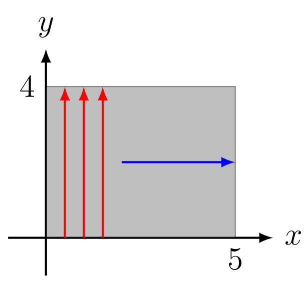
sofern der Grenzwert rechts existiert.

Beispiel (a)

$f(x,y) = x + y$

$G = \{(x,y) \in \mathbb{R}^2 \colon 0 \leq x \leq 5, 0 \leq y \leq 4\}$

Gebiet:



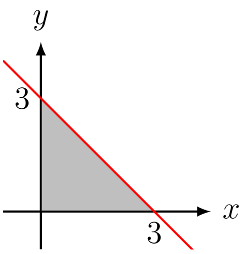
$$\begin{aligned} V &= \iint_G (x+y)\mathrm{d}G \\ &= \int_0^5 \int_0^4 (x+y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_0^5 \left[xy + \frac{1}{2}y^2\right]_0^4 \mathrm{d}x \\ &= \int_0^5 (4x+8)\,\mathrm{d}x \\ &= \left[2x^2 + 8x\right]_0^5 = (50+40) - (0+0) = 90 \end{aligned}$$

Beispiel (b)

$f(x, y) = 2xy$

$G = \{(x, y) \in \mathbb{R}^2 \colon 0 \leq x \leq 3, 0 \leq y \leq 3 - x\}$

Gebiet:

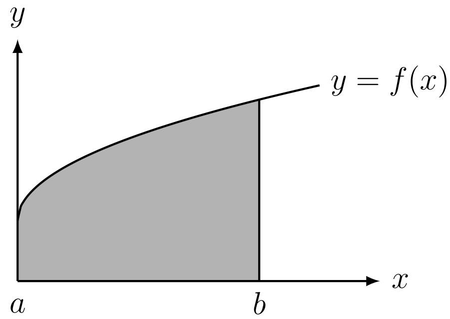


$$
\begin{aligned}
V &= \iint_G 2xy \,\mathrm{d}G \\
&= \int_0^3 \int_0^{3-x} 2xy \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^3 \left[xy^2\right]_0^{3-x} \mathrm{d}x \\
&= \int_0^3 x(3-x)^2 \,\mathrm{d}x \\
&= \int_0^3 \left(9x - 6x^2 + x^3\right) \mathrm{d}x \\
&= \left[4.5x^2 - 2x^3 + 0.25x^4\right]_0^3 \\
&= 6.75
\end{aligned}
$$

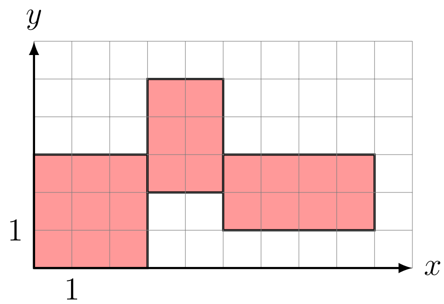
Flächenschwerpunkt ebener Flächen

Gegeben: ebener flächenhafter Körper mit konstanter Dicke h und homogener Dichte.

Gesucht: Flächenschwerpunkt der Grundfläche ($h \to 0$)



Zur Erinnerung:



$$x_S = \frac{x_1 \cdot A_1 + x_2 \cdot A_2 + x_3 A_3}{A_1 + A_2 + A_3}$$

$$y_S = \frac{y_1 \cdot A_1 + y_2 \cdot A_2 + y_3 A_3}{A_1 + A_2 + A_3}$$

$$x_s = \frac{\iint_A x \,\mathrm{d}A}{\iint_A \mathrm{d}A} = \frac{\int_a^b \int_0^{f(x)} x \,\mathrm{d}y \,\mathrm{d}x}{\int_a^b \int_0^{f(x)} 1 \,\mathrm{d}y \,\mathrm{d}x} = \frac{\int_a^b x \big[y\big]_0^{f(x)} \mathrm{d}x}{\int_a^b \big[y\big]_0^{f(x)} \mathrm{d}x}$$

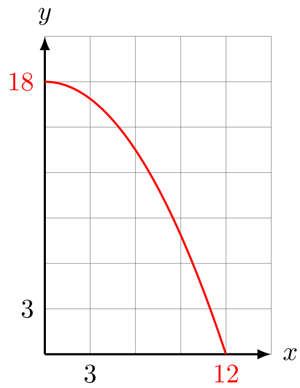
$$= \frac{\int_a^b x f(x) \,\mathrm{d}x}{\int_a^b f(x) \,\mathrm{d}x}$$

$$y_s = \frac{\iint_A y \,\mathrm{d}A}{\iint_A \mathrm{d}A} = \frac{\int_a^b \int_0^{f(x)} y \,\mathrm{d}y \,\mathrm{d}x}{\int_a^b \int_0^{f(x)} 1 \,\mathrm{d}y \,\mathrm{d}x} = \frac{\int_a^b \big[\frac{1}{2} y^2\big]_0^{f(x)} \mathrm{d}x}{\int_a^b \big[y\big]_0^{f(x)} \mathrm{d}x}$$

$$= \frac{\int_a^b \big(f(x)\big)^2 \,\mathrm{d}x}{2 \int_a^b f(x) \,\mathrm{d}x}$$

Beispiel (c)

Berechne den Schwerpunkt der Fläche, die von dem Graphen der Funktion $f(x) = 18 - \frac{1}{8}x^2$ und den positiven Koordinatenachsen eingeschlossen wird.



$$A = \int_0^{12} \left(18 - \frac{1}{8}x^2\right) \mathrm{d}x = \left[18x - \frac{1}{24}x^3\right]_0^{12} = 144$$

$$x_S = \frac{1}{144} \int_0^{12} x \left(18 - \frac{1}{2}x^2\right) \mathrm{d}x = \frac{1}{144} \int_0^{12} \left(18x - \frac{1}{8}x^3\right) \mathrm{d}x$$

$$= \frac{1}{144} \left[9x^2 - \frac{1}{32}x^3\right] = \frac{648}{144} = 4.5$$

$$y_S = \frac{1}{288} \int_0^{12} \left(18 - \frac{1}{2}x^2\right)^2 \mathrm{d}x = \frac{1}{288} \int_0^{12} \left(324 - \frac{9}{2}x^2 + \frac{1}{64}x^4\right) \mathrm{d}x$$

$$= \frac{1}{288} \left[324x - \frac{3}{2}x^3 + \frac{1}{320}x^5\right]_0^{12} = 7.2$$