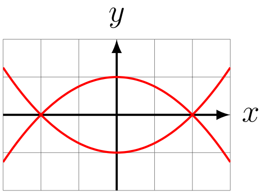
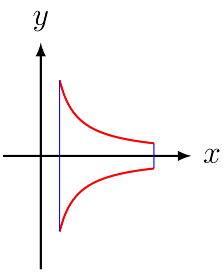
Aufgabe 7.1



Nullstellen: $\frac{1}{4}(x^2-4)=0 \quad \Rightarrow \quad x=\pm 2$

$$V = 2\pi \int_0^2 \left[\frac{1}{4}(x^2-4)\right]^2 \mathrm{d}x$$
$$= \frac{1}{8}\pi \int_0^2 \left(x^4-4x^2+16\right) \mathrm{d}x$$
$$= \frac{1}{8}\pi \left[\frac{1}{5}x^5-\frac{4}{3}x^3+16x\right]_0^2 = \frac{32}{15}\pi$$

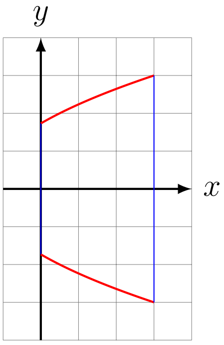
Aufgabe 7.2



$$V = \pi \int_a^b \left(x^{-1}\right)^2 \mathrm{d}x = \pi \int_a^b \left(x^{-2}\right) \mathrm{d}x = \pi \left[-x^{-1}\right]_a^b$$
$$= \pi\left(-\frac{1}{b}+\frac{1}{a}\right) = \pi\left(\frac{1}{a}-\frac{1}{b}\right)$$

Aufgabe 7.3

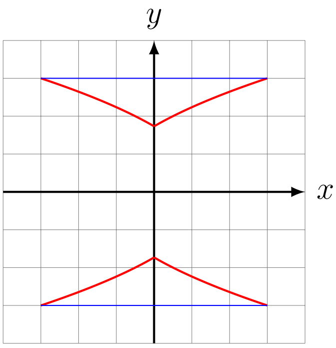
(a) $y^2 = 2x + 3 \quad \Leftrightarrow \quad y = \pm\sqrt{2x+3}$



$$V_x = \pi \int_1^3 (2x+3)\,\mathrm{d}x = \pi \int_1^3 \left(x^2 + 2x + 1\right) \mathrm{d}x$$

$$= \pi \left[x^2 + 2x + 1\right]_1^3 = 18\pi$$

(b) $y^2 = 2x + 3 \quad \Leftrightarrow \quad x = \frac{1}{2}\left(y^2 - 3\right)$



$$V_y = 2\pi \int_{\sqrt{3}}^3 \frac{1}{4}(y^2-3)^2\,\mathrm{d}y = \frac{\pi}{2} \int_{\sqrt{3}}^3 \left(y^4 - 6y^2 + 9\right) \mathrm{d}y$$

$$= \frac{\pi}{2} \left[\frac{1}{5}y^5 - 2y^3 + 9y\right]_{\sqrt{3}}^3$$

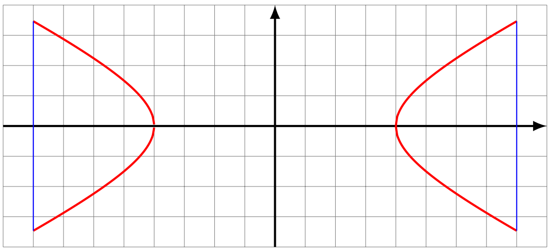
$$= \frac{\pi}{2} \left[\left(\frac{243}{5} - 54 + 27\right) - \left(\frac{9\sqrt{3}}{5} - 6\sqrt{3} + 9\sqrt{3}\right)\right]$$

$$= \frac{\pi}{2} \left[\frac{108}{5} - \frac{24\sqrt{3}}{5}\right]$$

$$= \frac{\pi}{5}\left(54 - 12\sqrt{3}\right) = \frac{6\pi}{5}\left(9 - 2\sqrt{3}\right)$$

Aufgabe 7.4

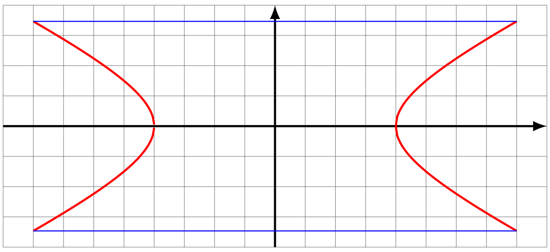
(a) $x^2 - 4y^2 = 16 \quad \Leftrightarrow \quad y^2 = \frac{1}{4}(x^2 - 16) \quad (4 \le x \le 8)$



$$V_x = 2\pi \int_4^8 \frac{1}{4}(x^2 - 16)\,\mathrm{d}x = \frac{\pi}{2}\left[\frac{1}{3}x^3 - 16x\right]_4^8$$

$$= \frac{\pi}{2}\left[\left(\frac{512}{3} - 128\right) - \left(\frac{64}{3} - 64\right)\right] = \frac{\pi}{2} \cdot \frac{256}{3}\pi = \frac{128}{3}\pi$$

(b) $x^2 - 4y^2 = 16 \quad \Leftrightarrow \quad x^2 = \sqrt{16 + 4y^2}$

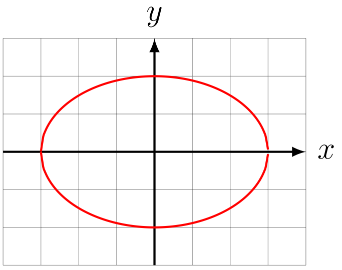


$$V_x = 2\pi \int_0^{2\sqrt{3}} (16 + 4y^2)\,\mathrm{d}y = 2\pi\left[16y + \frac{4}{3}y^3\right]_0^{2\sqrt{3}}$$

$$= 2\pi\left[32\sqrt{3} + 32\sqrt{3}\right] = 128\sqrt{3}\pi$$

Aufgabe 7.5

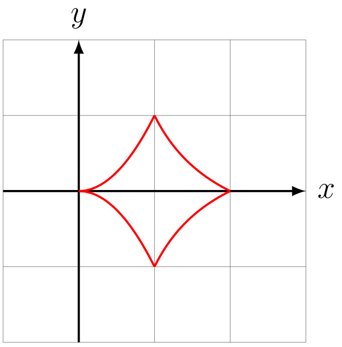
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad y^2 = b^2 - \frac{b^2}{a^2}x^2$$



$$V_x = 2\pi \int_0^a \left(b^2 - \frac{b^2}{a^2}x^2\right)\,\mathrm{d}x = 2b^2\pi \left[x - \frac{1}{3a^2}x^3\right]_0^a$$

$$= 2b^2\pi \left[a - \frac{1}{3}a\right] = \frac{4}{3}ab^2\pi$$

Aufgabe 7.6



Schnittstelle:

$$\begin{aligned} x^2 &= 2/x - 1 \\ x^3 &= 2 - x \\ x^3 + x - 2 &= 0 \\ x &= 1 \quad \text{(einzige reelle Nullstelle)} \end{aligned}$$

$$V_1 = \pi \int_0^1 (x^2)^2\,\mathrm{d}x = \pi \int_0^1 x^4\,\mathrm{d}x = \pi\left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}\pi$$

$$V_2 = \pi \int_1^2 (2x^{-1} - 1)^2\,\mathrm{d}x = \pi \int_1^2 \left(4x^{-2} - 4x^{-1} + 1\right)\mathrm{d}x$$

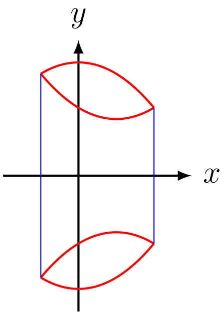
$$= \pi\left[-4x^{-1} - 4\ln(x) + x\right]_1^2$$

$$= \pi\big[(-2 - 4\ln(2) + 2) - (-4 - 0 + 1)\big]$$

$$= \pi\big[3 - 4\ln(2)\big]$$

$$V = V_1 + V_2 = \pi\big[3.2 - 4\ln(2)\big]$$

Aufgabe 7.7



Schnittstellen: $f(x) = g(x)$

$$x^2 - 2x + 6 = -x^2 + 10$$
$$2x^2 - 2x - 4 = 0$$
$$x^2 - x - 2 = 0$$
$$x_1 = -1$$
$$x_2 = 2$$

Äusseres Volumen:

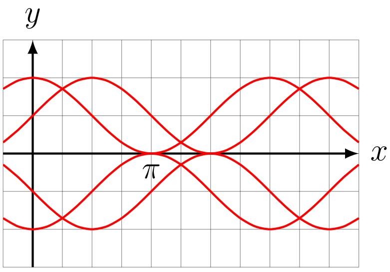
$$V_1 = \pi \int_{-1}^{2} \left(10 - x^2\right)^2 \mathrm{d}x$$
$$= \pi \int_{-1}^{2} \left(100 - 20x^2 + x^4\right) \mathrm{d}x$$
$$= \pi \left[100x - \frac{20}{3}x^3 + \frac{1}{5}x^5\right]_{-1}^{2} = \frac{1233}{5}\pi$$

Inneres Volumen:

$$V_2 = \pi \int_{-1}^{2} \left(x^2 - 2x + 6\right)^2 \mathrm{d}x$$
$$= \pi \int_{-1}^{2} \left(x^4 + 4x^2 + 36 - 4x^3 - 24x + 12x^2\right) \mathrm{d}x$$
$$= \pi \int_{-1}^{2} \left(x^4 - 4x^3 + 16x^2 - 24x + 36\right) \mathrm{d}x$$
$$= \pi \left[\frac{1}{5}x^5 - x^4 + \frac{16}{3}x^3 - 12x^2 + 36x\right]_{-1}^{2} = \frac{558}{5}\pi$$

Differenz: $V = V_2 - V_1 = \frac{1233}{5}\pi - \frac{558}{5}\pi = \frac{675}{5}\pi = 135\pi$

Aufgabe 7.8



Schnittstellen: $1 + \sin(x) = 1 + \cos(x)$

$$\sin(x) = \cos(x)$$

$$\frac{\sin(x)}{\cos(x)} = 1$$

$$\tan(x) = 1$$

$$x_k = \frac{\pi}{4} + k\pi$$

$$V_2 = \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \sin x)^2 \,\mathrm{d}x$$

$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(1 + 2\sin x + \sin^2 x\right) \mathrm{d}x$$

$$= \pi \left[x - 2\cos x + \frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \pi \left[\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \pi \left[\frac{3}{2} + 2\sqrt{2}\right]$$

$$V_1 = \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} (1 + \cos x)^2 \,\mathrm{d}x$$

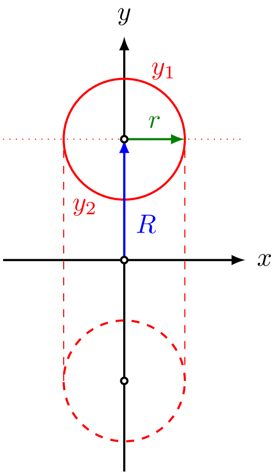
$$= \pi \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \left(1 + 2\cos x + \cos^2 x\right) \mathrm{d}x$$

$$= \pi \left[x - 2\cos x + \frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= \pi \left[\frac{3}{2} - 2\sqrt{2}\right]$$

$V = V_2 - V_1 = 4\sqrt{2}\pi$

Aufgabe 7.9



$y_1 = R + \sqrt{r^2 - x^2}$ oberer Halbkreis

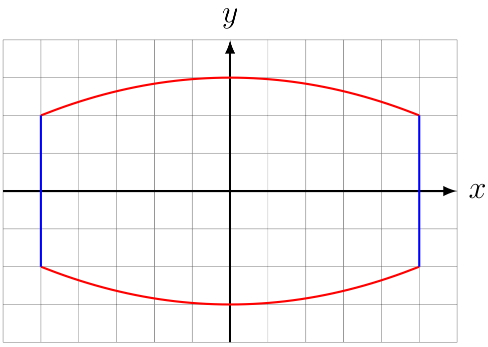
$y_2 = R - \sqrt{r^2 - x^2}$ unterer Halbkreis

$$V_1 = 2\pi \int_0^r \left(R + \sqrt{r^2 - x^2}\right)^2 \mathrm{d}x$$

$$V_2 = 2\pi \int_0^r \left(R - \sqrt{r^2 - x^2}\right)^2 \mathrm{d}x$$

$$\begin{aligned} V &= V_1 - V_2 \\ &= 2\pi \int_0^r \left[\left(R^2 + 2R\sqrt{*} + \sqrt{*}^2\right) - \left(R^2 - 2R\sqrt{*} + \sqrt{*}^2\right)\right] \mathrm{d}x \\ &= 8\pi R \int_0^r \sqrt{r^2 - x^2}\,\mathrm{d}x \\ &= 8\pi R \left[\frac{x}{2}\sqrt{r^2 - x^2} + \frac{r^2}{2} \arcsin\frac{x}{r}\right]_0^r \\ &= 8\pi R \left[\left(0 + \frac{r^2}{2}\arcsin 1\right) - \left(0 + \frac{r^2}{2}\arcsin 0\right)\right] \\ &= 2\pi R r^2 \end{aligned}$$

Aufgabe 7.10



$f(x) = ax^2 + b$

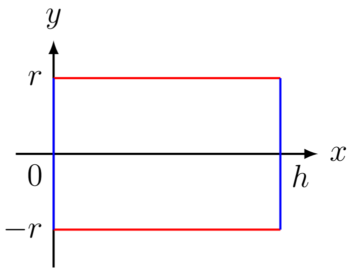
$f(0) = 3 = b$

$f(5) = 2 = 25a + 3 \quad \Rightarrow \quad a = -\dfrac{1}{25}$

$f(x) = 3 - \dfrac{1}{25}x^2$

$$\begin{aligned} V &= 2\pi \int_0^5 \left(3 - \frac{1}{25}x^2\right)^2 \,\mathrm{d}x \\ &= 2\pi \int_0^5 \left(9 - \frac{6}{25}x^2 + \frac{1}{625}x^4\right) \,\mathrm{d}x \\ &= 2\pi \left[9x - \frac{2}{25}x^3 + \frac{1}{3125}x^5\right]_0^5 \\ &= 2\pi[45 - 10 + 1] = 72\pi\,\mathrm{dm}^3 \approx 226.19\,\text{Liter} \end{aligned}$$

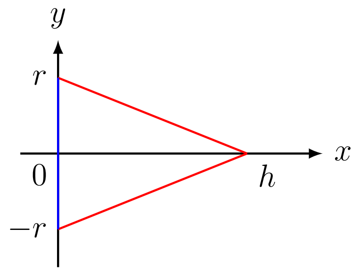
Aufgabe 7.11



$f(x) = r \quad \Rightarrow \quad f'(x) = 0$

$$M = 2\pi \int_0^h r\sqrt{1+0^2}\,\mathrm{d}x = 2\pi \int_0^h r\,\mathrm{d}x = 2\pi r\big[x\big]_0^h = 2\pi r h$$

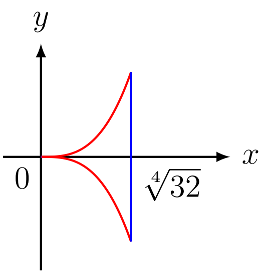
Aufgabe 7.12



$$f(x) = -\frac{r}{h}x + r \quad \Rightarrow \quad f'(x) = -\frac{r}{h}$$

$$\begin{aligned} M &= 2\pi \int_0^h \left(r - \frac{r}{h}x\right) \sqrt{1 + \left(-\frac{r}{h}\right)^2} \, \mathrm{d}x \\ &= 2\pi \sqrt{\frac{h^2 + r^2}{h^2}} \int_0^h \left(r - \frac{r}{h}x\right) \, \mathrm{d}x \\ &= 2\pi \sqrt{\frac{m^2}{h^2}} \left[rx - \frac{r}{2h}x^2\right]_0^h \\ &= 2\pi \frac{m}{h} \left(rh - \frac{r}{2h}h^2\right) \\ &= 2\pi \frac{m}{h} \cdot \frac{rh}{2} = \pi r m \end{aligned}$$

Aufgabe 7.13



$$f(x) = \frac{1}{6}x^3 \quad \Rightarrow \quad f'(x) = \frac{1}{2}x^2$$

$$M = 2\pi \int_0^{\sqrt[4]{32}} \frac{1}{6}x^3 \sqrt{1 + \frac{1}{4}x^4} \, \mathrm{d}x = \ldots$$

Substitution: $u = 1 + \frac{1}{4}x^4$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = x^3 \quad \Rightarrow \quad \mathrm{d}x = \frac{1}{x^3} \, \mathrm{d}u$$

Grenzen: $u(0) = 1 + 0 = 1$

$$u\left(\sqrt[4]{32}\right) = 1 + 8 = 9$$

$$
\begin{aligned}
\ldots &= 2\pi \int_1^9 \frac{1}{6} x^3 \sqrt{u} \frac{1}{x^3} \, \mathrm{d}u \\
&= \frac{1}{3}\pi \int_1^9 \sqrt{u} \, \mathrm{d}u \\
&= \frac{2}{9}\pi \big[u^{3/2}\big]_1^9 = \frac{2}{9}\pi(27-1) = \frac{52}{9}\pi
\end{aligned}
$$