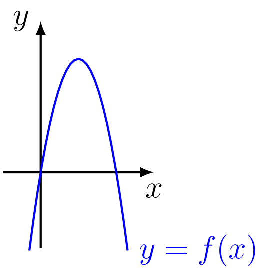
Aufgabe 6.1

Skizze:



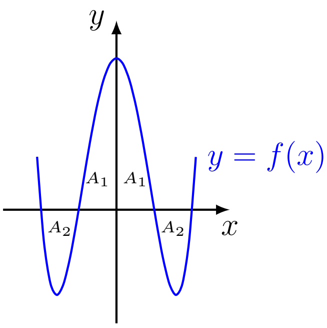
Nullstellen:

$$\begin{aligned} 6x - 3x^2 &= 0 \\ 3x(2-x) &= 0 \\ x_1 &= 0 \\ x_2 &= 2 \end{aligned}$$

$$\begin{aligned} A &= \int_0^2 (6x - 3x^2)\,\mathrm{d}x = \left[3x^2 - x^3\right]_0^2 \\ &= (12-8) - (0-0) = 4\,\mathrm{FE} \end{aligned}$$

Aufgabe 6.2

Skizze: (Ordinatensymmetrie)



Nullstellen:

$$\begin{aligned} x^4 - 5x^2 + 4 &= 0 \\ (x^2-4)(x^2-1) &= 0 \\ (x-2)(x+2)(x-1)(x+1) &= 0 \\ x_1 &= -2 \\ x_2 &= -1 \\ x_3 &= 1 \\ x_4 &= 2 \end{aligned}$$

$$\begin{aligned} A_1 &= \int_0^1 (x^4 - 5x^2 + 4)\,\mathrm{d}x = \left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 4x\right]_0^1 \\ &= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - (0 - 0 + 0) = \tfrac{3}{15} - \tfrac{25}{15} + \tfrac{60}{15} = \tfrac{38}{15} \end{aligned}$$

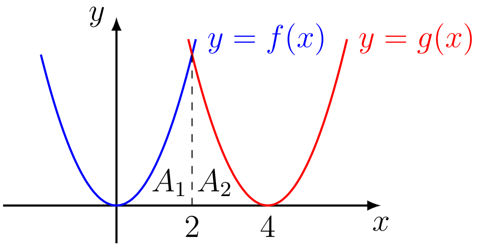
$$\begin{aligned} A_2 &= \int_2^1 (x^5 - 5x^2 + 4)\,\mathrm{d}x = \left[\tfrac{1}{5}x^5 - \tfrac{5}{3}x^3 + 4x\right]_2^1 \\ &= \left(\tfrac{1}{5} - \tfrac{5}{3} + 4\right) - \left(\tfrac{32}{5} - \tfrac{40}{3} + 8\right) \\ &= -\tfrac{31}{5} + \tfrac{35}{3} - 4 = -\tfrac{93}{15} + \tfrac{175}{15} - \tfrac{60}{15} = \tfrac{22}{15} \end{aligned}$$

Aufgrund der Symmetrie gilt:

$$A_{\text{Total}} = 2 \cdot (A_1 + A_2) = 2 \cdot \left(\tfrac{38}{15} + \tfrac{22}{15}\right) = 2 \cdot \tfrac{60}{15} = 2 \cdot 4 = 8\,\text{FE}$$

Aufgabe 6.3

Skizze:



Schnittstelle(n):

$x^2 = (x-4)^2$
$x^2 = x^2 - 8x + 16$
$0 = -8x + 16$
$x = 2$

Nullstelle von g:

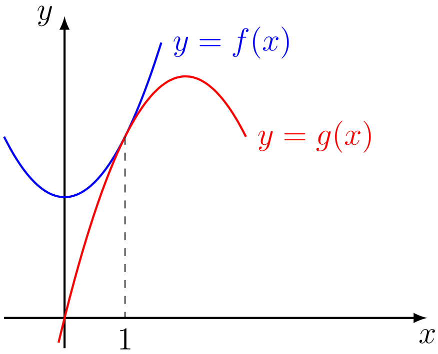
$(x-4)^2 = 0 \Rightarrow x = 4$

$$A_1 = \int_0^2 x^2 \,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3} - 0 = \frac{8}{3}$$

Aufgrund der Symmetrie gilt: $A_1 = A_2$ und damit: $A = 2 \cdot A_1 = \dfrac{16}{3}$ FE

Aufgabe 6.4

Skizze:



Berührstelle:

$$\begin{aligned} 2 + x^2 &= 4x - x^2 \\ 2x^2 - 4x + 2 &= 0 \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ x &= 1 \end{aligned}$$

Nachweis der Berührung: (gleiche Steigung bei $x = 1$)

$$f'(x) = 2x \quad \Rightarrow \quad f'(1) = 2$$

$$g'(x) = 4 - 2x \quad \Rightarrow \quad g'(1) = 2$$

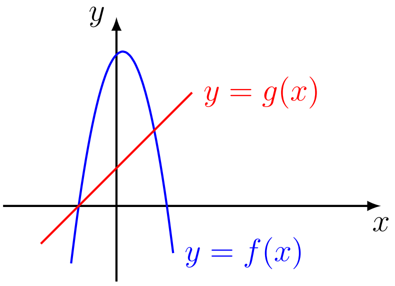
$$A_f = \int_0^1 \left(2 + x^2\right) \mathrm{d}x = \left[2x + \frac{1}{3}x^3\right]_0^1 = 2 + \frac{1}{3} = \frac{7}{3}$$

$$A_g = \int_0^1 \left(4x - x^2\right) \mathrm{d}x = \left[2x^2 - \frac{1}{3}x^3\right]_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$$

$$A_{\text{Total}} = A_f - A_g = \frac{7}{3} - \frac{5}{3} = \frac{2}{3}\,\text{FE}$$

Aufgabe 6.5

Skizze:

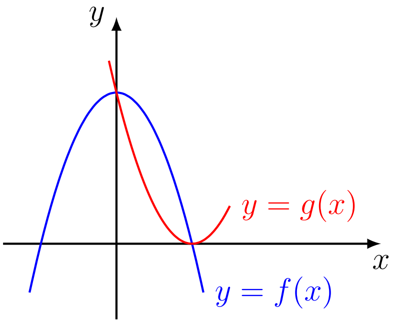


Schnittstellen: $$\begin{aligned} -3x^2 + x + 4 &= x + 1 \\ 3 &= 3x^2 \\ x^2 &= 1 \\ x_1 &= -1 \\ x_2 &= 1 \end{aligned}$$

$$\begin{aligned} A &= \int_{-1}^{1} [f(x) - g(x)]\,\mathrm{d}x = \int_{-1}^{1} \big(-3x^2 + x + 4 - (x+1)\big)\,\mathrm{d}x \\ &= \int_{-1}^{1} \big(-3x^2 + x + 4 - x - 1\big)\,\mathrm{d}x \\ &= \int_{-1}^{1} \big(-3x^2 + 3\big)\,\mathrm{d}x = \big[-x^3 + 3x\big]_{-1}^{1} \\ &= (-1+3) - (1-3) = 4\,\mathrm{FE} \end{aligned}$$

Aufgabe 6.6

Skizze:



Schnittstellen: $4 - x^2 = x^2 - 4x + 4$

$$0 = 2x^2 - 4x = 2x(x-2)$$
$$x_1 = 0$$
$$x_2 = 2$$

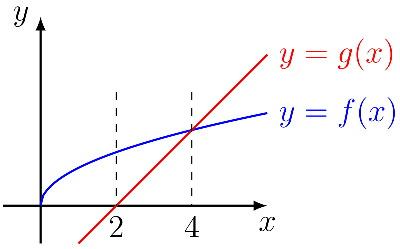
$$A = \int_0^2 \big[f(x) - g(x)\big]\,\mathrm{d}x = \int_0^2 \big(4 - x^2 - x^2 + 4x - 4\big)\,\mathrm{d}x$$

$$= \int_0^2 \big(-2x^2 + 4x\big)\,\mathrm{d}x = \big[-\frac{2}{3}x^3 + 2x^2\big]_0^2$$

$$= \left(-\frac{16}{3} + 8\right) - (0+0) = \frac{8}{3}\,\mathrm{FE}$$

Aufgabe 6.7

Skizze:



Schnittstellen:

$$\begin{aligned}
\sqrt{x} &= x - 2 \quad ||\,^2 \\
x &= (x-2)^2 = x^2 - 4x + 4 \quad || - x \\
0 &= x^2 - 5x + 4 \\
0 &= (x-1)(x-4) \\
x_1 &= -1 \quad (\text{Probe: } 1 = -1 \Rightarrow \text{Scheinlösung}) \\
x_2 &= 4 \quad (\text{Probe: } 2 = 2 \Rightarrow \text{ok})
\end{aligned}$$

Nullstelle von g: $x - 2 = 0$

$$x_1 = 2$$

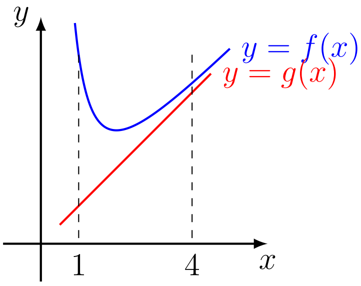
$$\begin{aligned}
A_1 &= \int_0^4 \sqrt{x}\,\mathrm{d}x = \frac{2}{3}\left[x^{3/2}\right]_0^4 = \frac{2}{3}\left[\sqrt{x^3}\right]_0^4 \\
&= \frac{2}{3}(\sqrt{64} - \sqrt{0}) = \frac{16}{3}
\end{aligned}$$

$$A_2 = \int_2^4 (x-2)\,\mathrm{d}x = \left[\frac{1}{2}x^2 - 2x\right]_2^4 = (8-8) - (2-4) = 2$$

$$A = A_1 - A_2 = \frac{16}{3} - 2 = \frac{10}{3}\,\text{FE}$$

Aufgabe 6.8

Skizze:



Gleichung der schiefen Asymptote:

$f(x) = x + 4/x^2 \approx x$ für grosse $|x| \quad \Rightarrow \quad g(x) = x$

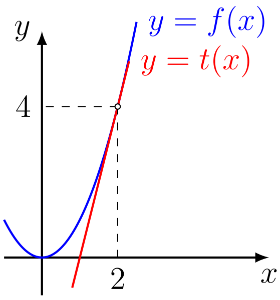
Schnittstellen: keine

$$A = \int_1^4 \big(f(x) - g(x)\big)\,\mathrm{d}x = \int_1^4 \big(x + 4x^{-2} + x\big)\,\mathrm{d}x = \int_1^4 4x^{-2}\,\mathrm{d}x$$

$$= \big[-4x^{-1}\big]_1^4 = -\frac{4}{4} - \left(-\frac{4}{1}\right) = 3\,\mathrm{FE}$$

Aufgabe 6.9

Skizze:



Gleichung der Tangente im Punkt $P(2,4)$:

$f'(x) = 2x \Rightarrow f'(2) = 2 \cdot 2 = 4 = m \quad$ (Steigung)

$y = mx + q$

$4 = 4 \cdot 2 + q$

$q = -4 \quad$ (Ordinatenabschnitt)

$\Rightarrow t\colon y = 4x - 4$

Nullstelle von f: $x^2 = 0 \Rightarrow x = 0$

Nullstelle von t: $0 = 4x - 4 \Rightarrow x = 1$

$$A_1 = \int_0^2 x^2 \,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_0^2 = \frac{8}{3}$$

$$A_2 = \frac{1 \cdot 4}{2} = 2 \text{ (Dreiecksfläche; Integrieren geht auch)}$$

$$A = A_1 - A_2 = \frac{8}{3} - 2 = \frac{2}{3}\,\text{FE}$$

Aufgabe 6.10

Setzt man die Koordinaten des Punkts $(2,2)$ in die Gleichung $y = ax^2$ ein, so erhält man $2 = a \cdot 4$ und daraus $a = \frac{1}{2}$.

Inhalt der Fläche *unter* dem Graphen:

$$\int_0^2 \frac{1}{2}x^2 \,\mathrm{d}x = \left[\frac{1}{6}x^3\right]_0^2 = \frac{8}{6} = \frac{4}{3}\,\text{FE}$$

Gesuchter Flächeninhalt: $A = 4 - \dfrac{4}{3} = \dfrac{8}{3}\,\text{FE}$

Aufgabe 6.11

$$A = \int_0^2 f(x)\ \mathrm{d}x = \left[\frac{1}{6}x^3 + \frac{1}{6}x^2 + x\right]_0^2 = 4$$

$$\frac{A}{2} = \int_0^m f(x)\ \mathrm{d}x$$

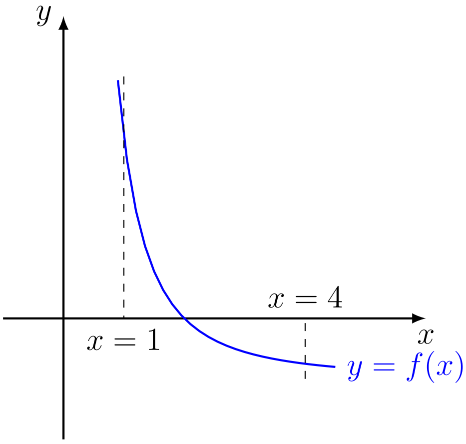
$$2 = \left[\frac{1}{6}x^3 + \frac{1}{6}x^2 + x\right]_0^m$$

$$2 = \frac{1}{6}m^3 + \frac{1}{6}m^2 + m$$

$$0 = \frac{1}{6}m^3 + \frac{1}{6}m^2 + m - 2 \Rightarrow m = 1.32$$

Aufgabe 6.12

Skizze:



Nullstelle:

$$\frac{4}{x^2} - 1 = 0 \quad || \cdot x^2 \neq 0$$

$$4 - x^2 = 0$$

$$x_1 = -2 \quad \text{(liegt nicht im Intervall)}$$

$$x_2 = 2$$

$$A_1 = \int_1^2 \left(4x^{-2} - 1\right) \mathrm{d}x = \left[-4x^{-1} - x\right]_1^2$$

$$= (-2 - 2) - (-4 - 1) = -4 + 5 = 1$$

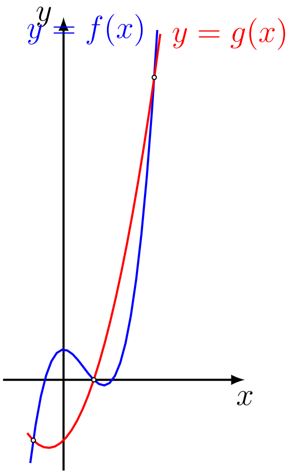
$$A_2 = \int_4^2 \left(4x^{-2} - 1\right) \mathrm{d}x = \left[-4x^{-1} - x\right]_4^2$$

$$= (-2 - 2) - (-1 - 4) = -4 + 5 = 1$$

$$A = A_1 + A_2 = 1 + 1 = 2\,\mathrm{FE}$$

Aufgabe 6.13

Skizze:



Schnittstellen:

$$x^3 - 2x^2 + 1 = x^2 + x - 2 \quad || - x^2 - x + 2$$

$$x^3 - 3x^2 - x + 3 = 0$$

$$x_1 = -1$$

$$x_2 = 1$$

$$x_3 = 3$$

$$A_1 = \int_{-1}^{1} \left(x^3 - 2x^2 + 1 - (x^2 + x - 2)\right) \mathrm{d}x$$

$$= \int_{-1}^{1} \left(x^3 - 3x^2 - x + 3\right) \mathrm{d}x = \left[\tfrac{1}{4}x^4 - x^3 - \tfrac{1}{2}x^2 + 3x\right]_{-1}^{1}$$

$$= \left(\tfrac{1}{4} - 1 - \tfrac{1}{2} + 3\right) - \left(\tfrac{1}{4} + 1 - \tfrac{1}{2} - 3\right) = -2 + 6 = 4$$

$$A_2 = \int_{3}^{1} \left[f(x) - g(x)\right] \mathrm{d}x$$

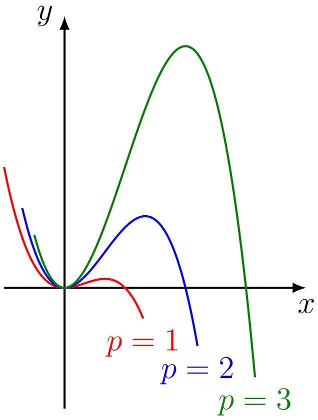
$$= \int_{3}^{1} \left(x^3 - 3x^2 - x + 3\right) \mathrm{d}x = \left[\tfrac{1}{4}x^4 - x^3 - \tfrac{1}{2}x^2 + 3x\right]_{3}^{1}$$

$$= \left(\tfrac{1}{4} - 1 - \tfrac{1}{2} + 3\right) - \left(\tfrac{81}{4} - 27 - \tfrac{9}{2} + 9\right) = 4$$

$$A = A_1 + A_2 = 4 + 4 = 8\,\mathrm{FE}$$

Aufgabe 6.14

Skizze:



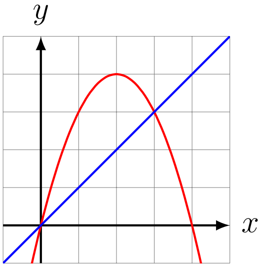
Offenbar haben alle Funktionen bei $x = 0$ eine doppelte Nullstelle und eine Nullstelle bei $x = p$, was durch folgende Rechnung zur Nullstellenbestimmung bestätigt wird:

$$
\begin{aligned}
px^2 - x^3 &= 0 \\
x^2(p - x) &= 0 \\
x_1 = x_2 &= 0 \\
x_3 &= p
\end{aligned}
$$

Mit dem gegebenen Flächeninhalt ($A = 4/3$) lässt sich folgende Gleichung aufstellen:

$$
\begin{aligned}
\int_0^p \left(px^2 - x^3\right) \mathrm{d}x &= \frac{4}{3} \\
\left[p \cdot \frac{1}{3}x^3 - \frac{1}{4}x^4\right]_0^p &= \frac{4}{3} \\
\left(p \cdot \frac{1}{3}p^3 - \frac{1}{4}p^4\right) - (0 - 0) &= \frac{4}{3} \\
\frac{1}{3}p^4 - \frac{1}{4}p^4 &= \frac{4}{3} \quad || \cdot 12 \\
4p^4 - 3p^4 &= 16 \\
p^4 &= 16 \\
p &= 2 \quad (p > 0)
\end{aligned}
$$

Aufgabe 6.15



$$
\begin{aligned}
G_f \cap G_g: \qquad 4x - x^2 &= mx \\
(4-m)x - x^2 &= 0 \\
x(4-m-x) &= 0 \\
x_1 &= 0 \\
x_2 &= 4-m
\end{aligned}
$$

$$A = \int_0^4 \left(4x - x^2\right)\,\mathrm{d}x = \left[2x^2 - \tfrac{1}{3}x^3\right]_0^4 = \left(32 - \tfrac{64}{3}\right) = \tfrac{32}{3}$$

$$
\begin{aligned}
\int_0^{4-m} \left(4x - x^2 - mx\right)\,\mathrm{d}x &= \tfrac{16}{3} \\
\left[2x^2 - \tfrac{1}{3}x^3 - \tfrac{m}{2}x^2\right]_0^{4-m} &= \tfrac{16}{3} \quad || \cdot 6 \\
\left[12x^2 - 2x^3 - 3mx^2\right]_0^{4-m} &= 32 \\
12(4-m)^2 - 2(4-m)^3 - 3m(4-m)^2 &= 32 \\
(4-m)^2\left[12 - 2(4-m) - 3m\right] &= 32 \\
(4-m)^2\left[4-m\right] &= 32 \\
(4-m)^3 &= 32 \\
a &= 4 - \sqrt[3]{32} \approx 0.8252
\end{aligned}
$$