Aufgabe 2.1

(a)
$$\begin{aligned} f(2+h) &= (2+h)^2 - 4(2+h) + 5 \\ &= 4 + 4h + h^2 - 8 - 4h + 5 \\ &= h^2 + 1 \end{aligned}$$

(b)
$$\begin{aligned} f(3+h) &= \frac{(3+h)+2}{(3+h)-3} = \frac{h+5}{h} \\ &= 1 + 5/h \end{aligned}$$

(c)
$$\begin{aligned} f(6+h) &= (6+h-4)^3 = (h+2)^3 \\ &= h^3 + 3 \cdot 2^1 \cdot h^2 + 3 \cdot 2^2 \cdot h^1 + 2^3 \\ &= h^3 + 6h^2 + 12h + 8 \end{aligned}$$

Aufgabe 2.2

$f(x) = x^2$; $x_0 = -3$

$$\begin{aligned} f'(-3) &= \lim_{h\to 0} \frac{f(-3+h) - f(-3)}{h} = \lim_{h\to 0} \frac{(-3+h)^2 - 3^2}{h} \\ &= \lim_{h\to 0} \frac{(-3+h)^2 - 9}{h} = \lim_{h\to 0} \frac{9 - 6h + h^2 - 9}{h} \\ &= \lim_{h\to 0} \frac{-6h + h^2}{h} = \lim_{h\to 0} \frac{h(-6+h)}{h} = \lim_{h\to 0}(-6+h) = -6 \end{aligned}$$

Aufgabe 2.3

$f(x) = 3x - 4$; $x_0 = 2$

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{3(2+h) - 4 - 2}{h} \\ &= \lim_{h\to 0} \frac{6 + 3h - 6}{h} = \lim_{h\to 0} \frac{3h}{h} = \lim_{h\to 0} 3 = 3 \end{aligned}$$

Aufgabe 2.4

$f(x) = x^3$; $x_0 = 2$

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{(2+h)^3 - 2^3}{h}$$
$$= \lim_{h \to 0} \frac{8 + 3 \cdot 2^2 \cdot h + 3 \cdot 2 \cdot h^2 + h^3 - 8}{h}$$
$$= \lim_{h \to 0} \frac{h(12 + 6h + h^2)}{h} = \lim_{h \to 0} (12 + 6h + h^2) = 12$$

Aufgabe 2.5

$f(x) = \frac{1}{x}$; $x_0 = 4$

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$
$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{4+h} - \frac{1}{4} \right]$$
$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{4}{4(4+h)} - \frac{(4+h)}{4(4+h)} \right] = \lim_{h \to 0} \left[\frac{1}{h} \cdot \frac{4 - (4+h)}{4(4+h)} \right]$$
$$= \lim_{h \to 0} \left[\frac{1}{h} \cdot \frac{-h}{4(4+h)} \right] = \lim_{h \to 0} \frac{-1}{4(4+h)} = -\frac{1}{16}$$

Aufgabe 2.6

$f(x) = \frac{1}{x+1}$; $x_0 = 1$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{1}{h} \left[\frac{1}{(1+h)+1} - \frac{1}{1+1} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{1}{2+h} - \frac{1}{2} \right] = \lim_{h\to 0} \frac{1}{h} \left[\frac{2-(2+h)}{2(2+h)} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{-h}{2(2+h)} \right] = \lim_{h\to 0} \frac{-1}{2(2+h)} = -\frac{1}{4}$$

Aufgabe 2.7

$f(x) = \frac{1}{x^2}$; $x_0 = 2$

$$f'(2) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \frac{1}{h} \left[\frac{1}{(2+h)^2} - \frac{1}{4} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{4-(2+h)^2}{4(2+h)^2} \right] = \lim_{h\to 0} \frac{1}{h} \left[\frac{4-(4+4h+h^2)}{4(2+h)^2} \right]$$

$$= \lim_{h\to 0} \frac{1}{h} \left[\frac{-4h-h^2}{4(2+h)^2} \right] = \lim_{h\to 0} \frac{-4-h}{4(2+h)^2} = -\frac{1}{4}$$

Aufgabe 2.8

$f(x) = \sqrt{x}$; $x_0 = 4$

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h} - \sqrt{4}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{4+h} - \sqrt{4}\right)\left(\sqrt{4+h} + \sqrt{4}\right)}{h\left(\sqrt{4+h} + \sqrt{4}\right)}$$

$$= \lim_{h\to 0} \frac{(4+h) - 4}{h\left(\sqrt{4+h} + \sqrt{4}\right)}$$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{4+h} + \sqrt{4}\right)} = \lim_{h\to 0} \frac{1}{\sqrt{4+h} + \sqrt{4}} = \frac{1}{4}$$

Aufgabe 2.9

$f(x) = \sqrt{2x};\ x_0 = 3$

$$
\begin{aligned}
f'(3) &= \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\sqrt{2(3+h)} - \sqrt{2\cdot 3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2h+6} - \sqrt{6}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{2h+6} - \sqrt{6}\right)\left(\sqrt{2h+6} + \sqrt{6}\right)}{h\left(\sqrt{2h+6} + \sqrt{6}\right)} \\
&= \lim_{h\to 0} \frac{(2h+6) - 6}{h\left(\sqrt{2h+6} + \sqrt{6}\right)} = \lim_{h\to 0} \frac{2h}{h\left(\sqrt{2h+6} + \sqrt{6}\right)} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2h+6} + \sqrt{6}} = \lim_{h\to 0} \frac{2}{2\sqrt{6}} = \frac{1}{\sqrt{6}}
\end{aligned}
$$

Aufgabe 2.10

$f(x) = \sqrt{x+3};\ x_0 = -2$

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{-2+h+3} - \sqrt{-2+3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{h+1} - 1}{h} = \lim_{h\to 0} \frac{\left(\sqrt{h+1} - 1\right)\left(\sqrt{h+1} + 1\right)}{h\left(\sqrt{h+1} + 1\right)} \\
&= \lim_{h\to 0} \frac{(h+1) - 1}{h\left(\sqrt{h+1} + 1\right)} = \lim_{h\to 0} \frac{h}{h\left(\sqrt{h+1} + 1\right)} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{h+1} + 1} = \frac{1}{2}
\end{aligned}
$$

Aufgabe 2.11

$f(x) = x^2 + 1;\ x_0 = -2$

$$
\begin{aligned}
f'(-2) &= \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{(-2+h)^2 + 1 - (4+1)}{h} \\
&= \lim_{h\to 0} \frac{4 - 4h + h^2 - 4}{h} = \lim_{h\to 0} \frac{h(h-4)}{h} \\
&= \lim_{h\to 0} (h-4) = -4
\end{aligned}
$$

$$
\begin{aligned}
y &= m_t x + q \\
d5 &= -4 \cdot (-2) + q \\
q &= -3 \quad \Rightarrow \quad t\colon y = -4x - 3
\end{aligned}
$$

Aufgabe 2.12

$f(x) = \sqrt{x-3};\ x = 4$

$$f'(4) = \lim_{h\to 0} \frac{f(4+h)-f(4)}{h} = \lim_{h\to 0} \frac{\sqrt{4+h-3}-1}{h}$$
$$= \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{(\sqrt{1+h}-1)(\sqrt{1+h}+1)}{h(\sqrt{1+h}+1)}$$
$$= \lim_{h\to 0} \frac{1+h-1}{h(\sqrt{1+h}+1)} = \lim_{h\to 0} \frac{h}{h(\sqrt{h+1}+1)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{h+1}+1} = \frac{1}{2}$$

$y = m_n\, x + q$

$1 = -2\cdot 4 + q$

$q = 9 \quad \Rightarrow \quad n\colon y = -2x + 9$

Aufgabe 2.13

$f(x) = \frac{1}{2x-1};\ x = 1$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{2(1+h)-1} - 1\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{1+2h} - 1\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{1-(1+2h)}{1+2h}\right]$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{-2h}{1+2h}\right] = \lim_{h\to 0} \frac{-2}{1+2h} = -2$$

$y = m_t\, x + q$

$1 = -2\cdot 1 + q$

$q = 3 \quad \Rightarrow \quad t\colon y = -2x + 3$

Aufgabe 2.14

$f(x) = \frac{1}{2}x^2 - x;\ x_0 = 1$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{2}(1+h)^2 - (1+h) + \frac{1}{2}}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{2} + h + \frac{1}{2}h^2 - 1 - h + \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{1}{2}h^2}{h}$$
$$= \lim_{h\to 0} \tfrac{1}{2}h = 0$$

Da die Tangente parallel zur x-Achse ist (Steigung: 0), muss die Normale senkrecht zur x-Achse verlaufen. Da $P(1, 0.5) \in G_f$, gilt: $n\colon x = 1$.

Aufgabe 2.15

$$f(x) = \frac{x-1}{x} = 1 - \frac{1}{x};\ x_0 = 2$$

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{1}{h}\left[f(2+h) - f(2)\right] = \lim_{h\to 0} \frac{1}{h}\left[1 - \frac{1}{2+h} - \frac{1}{2}\right] \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{1}{2} - \frac{1}{2+h}\right] = \lim_{h\to 0} \frac{1}{h}\left[\frac{2+h-2}{2(2+h)}\right] \\
&= \lim_{h\to 0} \frac{1}{2(2+h)} = \frac{1}{4}
\end{aligned}$$

$$y = m_t\, x + q$$

$$\frac{1}{2} = \frac{1}{4}\cdot 2 + q$$

$$q = 0 \quad \Rightarrow \quad t\colon y = \frac{1}{4}x$$

Aufgabe 2.16

$$f(x) = \sin(x);\ x = \tfrac{\pi}{4}$$

[trigonometrische Funktionen + Grenzwerte: Bogenmass]

$$\begin{aligned}
f'\left(\frac{\pi}{4}\right) &= \lim_{h\to 0} \frac{f\left(\frac{\pi}{4}+h\right) - f\left(\frac{\pi}{4}\right)}{h} = \lim_{h\to 0} \frac{\sin\left(\frac{\pi}{4}+h\right) - \sin\frac{\pi}{4}}{h} \\
&\overset{\text{S. 99}}{=} \lim_{h\to 0} \frac{\sin\frac{\pi}{4}\cos h + \cos\frac{\pi}{4}\sin h - \sin\frac{\pi}{4}}{h} \\
&= \lim_{h\to 0} \frac{\sin\frac{\pi}{4}(\cos h - 1) + \cos\frac{\pi}{4}\sin h}{h} \\
&= \sin\frac{\pi}{4}\lim_{h\to 0}\frac{\cos h - 1}{h} + \cos\frac{\pi}{4}\lim_{h\to 0}\frac{\sin h}{h} \\
&\overset{*}{=} \sin\frac{\pi}{4}\cdot 0 + \frac{\sqrt{2}}{2}\cdot 1 = \frac{\sqrt{2}}{2}
\end{aligned}$$

$$\begin{aligned}
*\colon\ \frac{\cos h - 1}{h} &= \frac{\cos h - \cos 0}{h} \overset{\text{S. 99}}{=} -\frac{2}{h}\sin\frac{h+0}{2}\sin\frac{h-0}{2} \\
&= -\frac{2}{h}\left(\sin\frac{h}{2}\right)^2 \qquad \text{substituiere } h' = \frac{h}{2} \Leftrightarrow h = 2h' \\
&= -\frac{2}{2h'}\cdot\left(\sin h'\right)^2 \\
&= -\sin h'\cdot\frac{\sin h'}{h'} \to 0\cdot 1 \quad \text{für } h'\to 0 \Leftrightarrow h\to 0
\end{aligned}$$

$$y = m_t\,x + q$$

$$\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} + q$$

$$q = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{\pi}{4} = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\sqrt{2}(4-\pi)}{8}$$

$$t\colon\, y = \frac{\sqrt{2}}{2}x + \frac{\sqrt{2}(4-\pi)}{8}$$

Aufgabe 2.17

$f(x) = \mathrm{e}^x$; $x = 1$

$$f'(1) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{\mathrm{e}^{1+h} - \mathrm{e}^1}{h}$$

$$= \lim_{h\to 0} \frac{\mathrm{e}^1\mathrm{e}^h - \mathrm{e}^1}{h} = \lim_{h\to 0} \frac{\mathrm{e}^1(\mathrm{e}^h - 1)}{h}$$

$$= \mathrm{e} \lim_{h\to 0} \frac{\mathrm{e}^h - 1}{h} = \ldots$$

$\mathrm{e}^x \overset{\text{Def.}}{=} \lim\limits_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ ist hier unpraktisch.

Alternative: $\mathrm{e}^x = \sum\limits_{k=0}^{\infty} \frac{1}{k!}x^k = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$

$= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \ldots$ (Nachweis: später)

$$\ldots = \mathrm{e} \lim_{h\to 0} \frac{\mathrm{e}^h - 1}{h}$$

$$= \mathrm{e} \lim_{h\to 0} \frac{1}{h}\left[\left(1 + h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 + \ldots\right) - 1\right]$$

$$= \mathrm{e} \lim_{h\to 0} \frac{1}{h}\left[h + \frac{1}{2!}h^2 + \frac{1}{3!}h^3 + \ldots\right]$$

$$= \mathrm{e} \lim_{h\to 0} \left[1 + \frac{1}{2!}h + \frac{1}{3!}h^2 + \ldots\right]$$

$$= \mathrm{e} \cdot 1 = \mathrm{e}$$

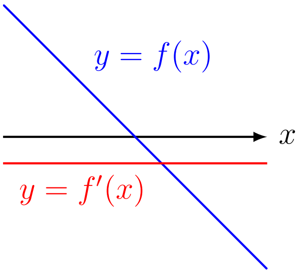
$$y = mx + q$$

$$\mathrm{e}^1 = \mathrm{e} \cdot 1 + q$$

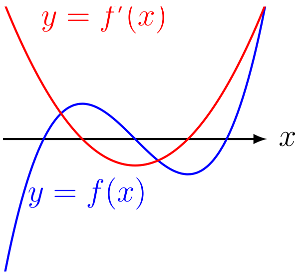
$$q = 0$$

$$t\colon\, y = \mathrm{e} \cdot x$$

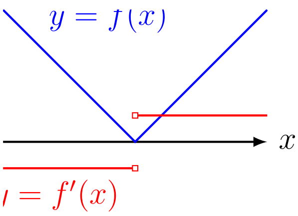
Aufgabe 2.18



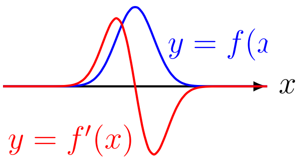
Aufgabe 2.19



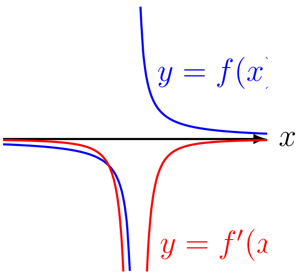
Aufgabe 2.20



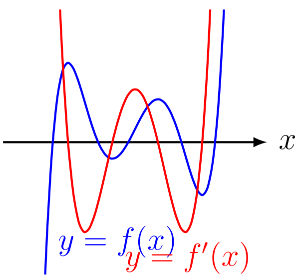
Aufgabe 2.21



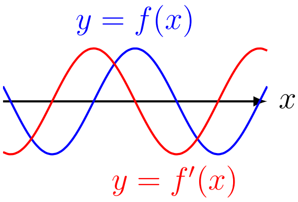
Aufgabe 2.22



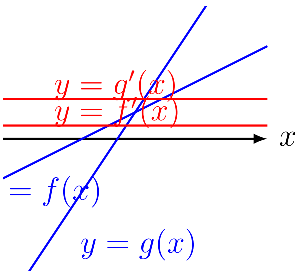
Aufgabe 2.23



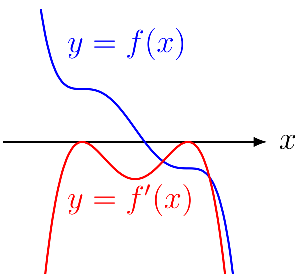
Aufgabe 2.24



Aufgabe 2.25



Aufgabe 2.26



Aufgabe 2.27